Obvious Dominance and Random Priority

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Abstract

We introduce a general class of simplicity concepts that vary the foresight abilities required of agents in extensive-form games, and use it to provide characterizations of simple mechanisms in social choice environments with and without transfers. We show that obvious strategy-proofness—an important simplicity concept included in our class—is characterized by clinch-or-pass games we call millipede games. Some millipede games are indeed simple and widely-used, though others may be complex, requiring significant foresight on the part of the agents, and are rarely observed. Weakening the foresight abilities assumed of the agents eliminates these complex millipede games, leaving monotonic games as the only simple games, a class which includes ascending auctions. As an application, we explain the widespread popularity of the well-known Random Priority mechanism by showing it is the unique mechanism that is efficient, fair, and simple to play.

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1 Introduction

Consider a group of agents who must come together to make a choice from some set of potential outcomes that will affect each of them. This can be modeled as having the agents play a “game”, taking turns choosing from sets of actions (possibly simultaneously), with the final outcome determined by the decisions made by all of the agents each time they were called to play. To ensure that the ultimate decision taken satisfies desirable normative properties (e.g., efficiency), the incentives given to the agents are crucial. The standard route taken in mechanism design is to appeal to the revelation principle and use a strategy-proof direct mechanism where agents are simply asked to report their private information, and it is always in their interest to do so truthfully. However, this is useful only to the extent the participants understand that a given mechanism is strategy-proof, and indeed, there is evidence many real-world agents do not tell the truth, even in strategy-proof mechanisms. In other words, strategy-proof mechanisms, while theoretically appealing, may not actually be easy for participants to play in practice.

What mechanisms, then, are actually “simple to play”? And further, what simple mechanisms satisfy other normatively desirable properties such as efficiency and fairness? We address these questions for a broad range of social choice environments both with and without transfers. We do so by introducing a general class of simplicity concepts that vary the foresight abilities required of agents in extensive-form imperfect-information games, and use them to provide characterizations of simple mechanisms for a wide range of settings, including such simple popular mechanisms as posted prices and Random Priority.

Our general approach relies on the idea that, rather than planning a strategy for any possible future point that may be reached in a game, a player plans for only these nodes (or information sets) of the game that he or she perceives as simple.\(^1\) For such a strategic plan to be dominant, the called for action needs to be unambiguously better than alternatives, irrespective of what happens at nodes that are not simple for the agent. As the game progress, the agent perception of which nodes are simple may change. Importantly, we allow for the possibility that the agent can change his or her strategic plan along the path of the game. The sets of nodes perceived as simple from the perspective of a given node is taken as a primitive of our definition, and the smaller (in an inclusion sense) are these sets of simple nodes, the stronger is the resulting simplicity concept.

One important special case covered by our approach is Li’s (2017b) obvious dominance.\(^2\)

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\(^1\)Savage (1954) wrestles with whether decision-makers should be modeled as “look before you leap” (create a full strategic plan for all possible future decisions one may face) or “you can cross that bridge when you come to it” (make choices as they arise). While standard strategic concepts of game theory formalize the former modeling option, our approach formalizes the latter.

\(^2\)Li (2017b) provides both a theoretical behavioral foundation for obvious dominance as well as experi-
A strategy in an imperfect-information extensive-form game to be *obviously dominant* if, whenever an agent is called to play, even the worst possible final outcome from following the action prescribed by the strategy is at least as good as the best possible outcome from taking any other action at the node in question, where the best and worst cases are determined by fixing the agent’s strategy and considering all possible strategies that could be played by her opponents in the future. Our general approach to simplicity captures obviously dominant strategies when agents perceive all of their own information sets as simple and all information sets of other agents as not simple. This highlights an important feature of obvious dominance, which is that it presumes that agents can perform demanding backward induction over their own future actions. As an example, consider chess: assuming that White can always force a win, any winning strategy of White is obviously dominant, and yet the strategic choices in chess are far from obvious.

To get a better understanding of what strategies are indeed simple, we also analyze more demanding concepts in our class, namely one-step-foresight dominance and strong obvious dominance. A strategic plan is *one-step-foresight dominant* if it is dominant for players who perceives as simple their current information set and only the first information sets at which they may be called to play of the continuation game; in other words, agents are able to plan at most one move ahead. For instance, in an ascending auction planning to stay in is dominant for such players as long as the current price is below their value because they can foresee the next round of bidding and they can always drop out at the next round.\(^3\)

A strategic plan is *strongly obviously dominant* if it is dominant for players who perceive as simple only their current information set. In other words, the strategy is strongly obviously dominant if, whenever an agent is called to play, even the worst possible final outcome from the prescribed action is at least as good as the best possible outcome from any other action, where what is possible may depend on all future actions, including actions by the agent’s future-self. Thus, strongly obviously dominant strategies are those that are weakly better than all alternative strategies even if the agent is concerned that she might tremble in the future or has time-inconsistent preferences.

For each of these three subclasses of our general simplicity construction we analyze which games are simple. In the case of obvious dominance we focus on the social choice environments without transfers, hence complementing Li (2017b) who focused on the case with transfers.\(^4\) We call the class of obviously dominant games in these environments *millipede*

\(^3\)Notice that our framework allows for the bidder’s strategic plan to be adjusted when the next round is actually reached.

\(^4\)Social choice problems without transfers are ubiquitous in the real-world, and examples include refugee
In a millipede game, each time an agent is called to move, she is presented with some subset outcomes that she can ‘clinch’, after which she leaves the game; she also may be given the opportunity to ‘pass’ and remain in the game, with the potential of being offered better clinching options in the future.\textsuperscript{5} If this agent passes, another agent is presented with an analogous choice, etc., until one of them eventually clinches. While some millipede games, such as serial dictatorships, are frequently encountered and are indeed simple to play, others are rarely observed in market-design practice, and their strategy-proofness is not necessarily immediately clear. In particular, similar to chess, some millipede games require agents to look far into the future and to perform potentially complicated backward induction reasoning.

We study one-step-foresight dominance in our general model that allows both environments with and without transfers. We establish that all such games can be implemented as monotonic games. Using the terminology of object allocation for expositional simplicity, a game is monotonic if, at any next move (if there is one), the player can either clinch all the payoffs that were clinchable previously, or, in the last move, the player can clinch any other still possible payoff. Ascending auctions provide an example of such a monotonicity: the only clinchable payoff is that associated with dropping out, except if the agent wins. We further show that in environments including single-good auctions and binary public good choice, any social choice rule that is implementable in obviously dominant strategies is also implementable in one-step-foresight dominant strategies.\textsuperscript{6} In object allocation with single-unit demand and no transfers, a game is monotonic if either the subset of objects she is offered to clinch grows larger each time she is called to move (in an inclusion sense), or she is offered everything that is still possible for her, and is never called to move again.

We also study strong obvious dominance also in our general model. We show that strongly obviously strategy-proof games do not require agents to look far into the future and perform lengthy backwards induction: in all such games, each agent has essentially at most one payoff-relevant move. Building on this insight, we show that all strongly obvious strategy-proof games can be implemented as sequential personalized price games in which each agent is offered, at most once, a choice from a menu of outcome-price pairs. If the menu has three or more undominated object-price pairs, then then agent’s final outcome is what they choose.

\textsuperscript{5}While there can be many clinching actions, there can be at most one passing action. In this aspect, millipedes resemble the centipede game of (Rosenthal, 1981).

\textsuperscript{6}This result builds on Li’s (2017b) characterization of obvious dominance implementation in these environments.
from the menu. If the menu has only two undominated pairs, then the final outcome might depend on other agents’ choices, but truthfully indicating the preferred object-price pair is the dominant choice. In this way, strong obvious dominance gives us a microfoundation for posted prices, a ubiquitous sale mechanism.

While most of our focus is on incentives and simplicity, there are other important criteria when designing a mechanism, such as efficiency and fairness. In the final section of the paper we study these normative criteria in the context of no-transfer allocation problems and show that only one mechanism satisfies all three requirements: Random Priority (also known as Random Serial Dictatorship). In Random Priority, first nature selects an ordering of agents, and then each agent moves in turn and chooses the favorite object among those that remain available given previous agents’ choices. This mechanism has a long history and is extensively used in a wide variety of practical allocation problems, including school choice, worker assignment, course allocation, and the allocation of public housing. The mechanism is well-known to have good efficiency, fairness, and incentive properties: it is Pareto efficient, it treats agents in a symmetric way, and it is obviously strategy-proof. However, it has until now remained unknown whether there are other such mechanisms, and if so, what explains the relative popularity of Random Priority over these alternatives. We show that there are none, thus resolving positively the quest to establish Random Priority as the unique mechanism with good incentive, efficiency, and fairness properties and thereby explaining its popularity in practical market design settings.

Our results build on the key contributions of Li (2017b), who formalized obvious strategy-proofness and established its desirability as an incentive property (see the discussion above). Our construction of the simplicity criteria—while being more general and allowing us to

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7 An object-price pair is dominated if the menu contains another pair with the same object and a lower price. In environments without transfers, strongly obviously strategy-proof and Pareto efficient games take the simple form of almost-sequential dictatorships, studied earlier in a different context by Pycia and Ünver (2016).

8 For an earlier microfoundation of posted prices, see Hagerty and Rogerson (1987). Armstrong (1996) shows that posted prices (combined with bundling) can achieve good revenues (for other analyses of revenues of posted price mechanisms see also e.g. Chawla et al. (2010) and Feldman et al. (2014)). Empirically, even on eBay, which began as an auction website, Einav et al. (2018) document a dramatic shift in the 2000s from auctions to posted prices as the predominant selling mechanism on the platform.

9 For discussion of efficiency and fairness see, e.g., Abdulkadiroğlu and Sönmez (1998), Bogomolnaia and Moulin (2001), Che and Kojima (2010), and Liu and Pycia (2011). Obvious strategy-proofness of Random Priority was established by Li (2017b).

10 In single-unit demand allocation with at most three agents and three objects, Bogomolnaia and Moulin (2001) proved that Random Priority is the unique mechanism that is strategy-proof, efficient, and symmetric. In markets in which each object is represented by many copies, Liu and Pycia (2011) and Pycia (2011) proved that Random Priority is the asymptotically unique mechanism that is symmetric, asymptotically strategy-proof, and asymptotically ordinally efficient. While these earlier results looked at either very small or very large markets, ours is the first characterization that holds for any number of agents and objects.
select more precisely simple mechanisms—is inspired by his work. Li’s work generated a substantive interest focused on his simplicity concept. Following up on Li’s work, but preceding ours, Ashlagi and Gonczarowski (2018) show that stable mechanisms such as Deferred Acceptance are not obviously strategy-proof, except in very restrictive environments where Deferred Acceptance simplifies to an obviously strategy-proof game with a ‘clinch or pass’ structure similar to simple millipede games (though they do not describe it in these terms). Other related papers include Troyan (2019), who studies obviously strategy-proof allocation via the popular Top Trading Cycles (TTC) mechanism, and provides a characterization of the priority structures under which TTC is OSP-implementable. Following our work, Arribillaga et al. (2017) characterize the voting rules that are obviously strategy-proof on the domain of single-peaked preferences and, in an additional result, in environments with two alternatives; Bade and Gonczarowski (2017) study obviously strategy-proof and efficient social choice rules in several environments. Mackenzie (2017) introduces the notion of a “round table mechanism” for OSP implementation and draws parallels with the standard Myerson-Riley revelation principle for direct mechanisms. There has been less work that goes beyond Li’s obvious dominance. Li (2017a) extends his ideas to ex post equilibrium context, while Zhang and Levin (2017a; 2017b) provide decision-theoretic foundations for obvious dominance and explore weaker incentive concepts.

More generally, our work also contributes to the understanding of limited foresight and limits on backward induction. Other work in this area—with very different approach from ours—includes Jehiel’s (1995; 2001) studies of limited foresight, Ke’s (2015) axiomatic approach to bounded horizon backward induction, as well as the rich literature on time-inconsistent preferences (e.g., Laibson (1997) and Gul and Pesendorfer (2001; 2004)). The paper also adds to our understanding of dominant incentives, efficiency, and fairness in settings with and without transfers. In settings with transfers, these questions were studied by e.g. Vickrey (1961), Clarke (1971), Groves (1973), Green and Laffont (1977), Holmstrom (1979), Dasgupta et al. (1979), and Hagerty and Rogerson (1987). In settings without transfers, in addition to Gibbard (1973, 1977) and Satterthwaite (1975) and the allocation papers mentioned above, the literature on mechanisms satisfying these key objectives includes Pápai (2000), Ehlers (2002) and Pycia and Unver (2016; 2017) who characterized efficient and group strategy-proof mechanisms in settings with single-unit demand, and Pápai (2001) and

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11 Li showed that the classic top trading cycles (TTC) mechanism of Shapley and Scarf (1974), in which each agent starts by owning exactly one object, is not obviously strategy-proof. Also of note is Loertscher and Marx (2015) who study environments with transfers and construct a prior-free obviously strategy-proof mechanism that becomes asymptotically optimal as the number of buyers and sellers grows.

12 This last paper builds on our work characterizing obvious dominance (released in 2016), but some of our characterization results on Random Priority (added in 2019) in turn build on their insights into Pareto efficient obviously strategy-proof mechanisms.

2 Model

2.1 Preferences

Let $\mathcal{N} = \{i_1, \ldots, i_N\}$ be a set of agents, and $\mathcal{X}$ a finite set of outcomes.\textsuperscript{14} Each agent has a preference ranking over outcomes, where, for any two $x, y \in \mathcal{X}$, we write $x \succeq_i y$ to denote that $x$ is weakly preferred to $y$. We allow for indifferences, and write $x \sim_i y$ if $x \succeq_i y$ and $y \succeq_i x$. The domain of preferences of agent $i \in \mathcal{N}$ is denoted $\mathcal{P}_i$. For any $\succeq_i$, we let $\succ_i$ denote the corresponding strict preference relation, where $x \succ_i y$ if $x \succeq_i y$ but not $y \succeq_i x$.

We will generally work with the strict preference relation $\succ_i$, which we refer to as an agent’s type.

We study both settings with and without transfers. The main assumption we make on the preference domains is that they are rich. Our formalization of richness takes as a primitive a dominance relation over over outcomes, denoted $\succeq_i$, where formally, $\succeq_i$ is a reflexive and transitive binary relation on $\mathcal{X}$. If $x \succeq_i y$ but not $y \succeq_i x$, then we write $x \succ_i y$. A preference ranking $\succeq_i$ is consistent with $\succeq_i$ if $x \succeq_i y$ implies that $x \succ_i y$ and $x \succ_i y$ implies that $x \succ_i y$.

Then, we say that $\mathcal{P}_i$ is rich if it contains all strict rankings of outcomes that are consistent with $\succeq_i$. We allow that different agents have different preference domains; that is different agents’ preference domains might be governed by different dominance relations, $\succeq_i$.

If $x \succeq_i y$ and $y \succeq_i x$ then we say that $x$ and $y$ are $\succeq_i$-equivalent. Any such $\succeq_i$ determines a partition of $\mathcal{X}$, which we refer to as an equivalence partition. We refer to each element of the equivalence partition as a payoff of the agent in question.\textsuperscript{15,16}

\textsuperscript{13}Pycia and Ünver (2016) characterized individually strategy-proof and Arrovian efficient mechanisms. For an analysis of these issues under additional feasibility constraints, see also Dur and Ünver (2015).

\textsuperscript{14}Assuming $\mathcal{X}$ is finite simplifies the exposition and it is satisfied in examples such as voting and the no-transfer allocation environments listed in the introduction. This assumption can be relaxed. For instance, our analysis goes through with no substantive changes if we allow infinite $\mathcal{X}$ endowed with a topology such that agents’ preferences are continuous in this topology and the relevant sets of outcomes are compact.

\textsuperscript{15}More formally, we can write $[x]_i = \{y \in \mathcal{X} : x \succeq_i y \text{ and } y \succeq_i x\}$ to represent the element of the partition that contains $x$. In the sequel, we will generally use just $x$ to refer to the entire partition element for an agent, where phrases such as “payoff $x$ obtains” means “some $y \in [x]_i$ obtains”.

\textsuperscript{16}We might slightly relax the richness assumption. For instance, in the presence of outside options we would say that a game is individually rational if each agent can obtain at least his outside option. To obtain the analogues of many of our results for individually rational games, it is sufficient to assume that the domain of each agent’s preferences satisfies the richness condition restricted to sets $X \subseteq \mathcal{X}$ that do not contain the...
Examples of rich domains without transfers include the canonical voting environment (where every agent can strictly rank all alternatives) and allocating a set of indivisible goods (where each agent cares only about the set of goods she receives). In the former case, the dominance relation $\succeq$ is the trivial relation in which $x \succeq y$ implies that $x = y$; equivalently, each agent partitions $X$ into $|X|$ singleton subsets and an agent’s type $\succ_i$ is a strict preference relation over $X$. Richness then implies that any strict ranking of the outcomes is possible for each agent. In the latter case, an outcome $x \in X$ describes the entire allocation to each of the agents, and each agent is indifferent over how goods she does not receive are assigned to others. This is captured by a dominance relation $\succeq_i$ of agent $i$ defined as follows: $x \succeq_i y$ if and only if agent $i$ receives the same set of goods in outcomes $x$ and $y$. Each element of agent $i$’s partition of $X$ can be identified with her own individual allocation, and her type $\succ_i$ can then be identified with a strict ranking of these allocations. Richness then implies that every strict ranking of $i$’s allocations is possible for each agent.

Examples of rich domains with transfers also include many standard social choice and allocation environments. For instance, consider a non-empty finite set of substantive outcomes $Y$ and a non-empty finite set $W \subseteq \mathbb{R}$ of possible transfers, and the set of outcomes $X$ that takes the form $X = Y \times W^N$. We use $w_i$ to denote the transfer (payment) made by agent $i$, and $w \equiv (w_i)_{i \in N}$ to denote an entire profile of transfers. Each agent $i$ prefers to pay less rather than more (keeping the substantive outcome the same) and is indifferent between any two outcomes that vary only in other agents’ transfers. This preference domain is given by the dominance relation $(y, (w_1, ..., w_n)) \succeq_i (y', (w'_1, ..., w'_n))$ if and only if $y = y'$ and $w_i \geq w'_i$. A related example obtains if the set of substantive outcomes $Y$ is given as the set of feasible allocations of objects to agents, and agents care only about the objects they obtain and their own payments. The preference domain is given by the dominance relation $((y_1, ..., y_n), (w_1, ..., w_n)) \succeq_i ((y'_1, ..., y'_n), (w'_1, ..., w'_n))$ if and only if $y_i = y'_i$ and $w_i \geq w'_i$.

While richness is a very flexible assumption, not all preference domains are rich. For instance, domains of single-peaked preferences are typically not rich.

When dealing with lotteries, we are agnostic as to how agents evaluate them, as long as the following property holds: an agent prefers lottery $\mu$ over $\nu$ if for any outcomes $x \in \text{supp} (\mu)$ and $y \in \text{supp} (\nu)$ this agent weakly prefers $x$ over $y$; the preference between $\mu$ and $\nu$ is strict if, additionally, at least one of the preferences between $x \in \text{supp} (\mu)$ and $y \in \text{supp} (\nu)$ is strict. This mild assumption is satisfied for expected utility agents; it is also satisfied for agents who prefer $\mu$ to $\nu$ as soon as $\mu$ first-order stochastically dominates $\nu$. 

outside option of this agent.
2.2 Mechanisms

To determine the outcome that will be implemented, the planner designs a game \( \Gamma \) for the agents to play. Formally, we consider imperfect-information, extensive-form games with perfect recall, which are defined in the standard way: there is a finite collection of partially ordered histories (sequences of moves), \( \mathcal{H} \). We use the notation \( h' \subseteq h \) to denote that \( h' \in \mathcal{H} \) is a subhistory of \( h \in \mathcal{H} \) (equivalently, \( h \) is a continuation history of \( h' \)), and write \( h \subset h' \) when \( h \subseteq h' \) but \( h \neq h' \). Terminal histories (those with no successors) will be denoted with bars, i.e., \( \bar{h} \). Each \( h \in \mathcal{H} \) is associated with an outcome in \( \mathcal{X} \), and agents receive payoffs at \( \bar{h} \) that are consistent with their preferences over outcomes \( \succ_i \). At every non-terminal history \( h \in \mathcal{H} \), one agent, denoted \( i_h \), is called to play and has a finite set of actions \( A(h) \) from which to choose. We write \( h' = (h, a) \) to denote the history \( h' \) that is reached by starting at history \( h \) and following the action \( a \in A(h) \). To avoid trivialities, we assume that no agent moves twice in a row and that \( |A(h)| > 1 \) for all non-terminal \( h \in \mathcal{H} \). To capture random mechanisms, we also allow for histories \( h \) at which a non-strategic agent, Nature, is called to move, and selects an action in \( A(h) \) according to some probability distribution.

The set of histories at which agent \( i \) moves is denoted \( \mathcal{H}_i = \{ h \in \mathcal{H} : i_h = i \} \). The set \( \mathcal{I}_i \) is a partition of \( \mathcal{H}_i \) into information sets, where, for any information set \( I \in \mathcal{I}_i \) and \( h, h' \in I \) and any subhistories \( \bar{h} \subseteq h \) and \( \bar{h}' \subseteq h' \) at which \( i \) moves, at least one of the following two symmetric conditions obtains: either (i) there is a history \( \bar{h}^* \subseteq \bar{h} \) such that \( \bar{h}^* \) and \( \bar{h}' \) are in the same information set, \( A(\bar{h}^*) = A(\bar{h}') \), and \( i \) makes the same move at \( \bar{h}^* \) and \( \bar{h}' \), or (ii) there is a history \( \bar{h}^* \subseteq \bar{h}' \) such that \( \bar{h}^* \) and \( \bar{h} \) are in the same information set, \( A(\bar{h}^*) = A(\bar{h}) \), and \( i \) makes the same move at \( \bar{h}^* \) and \( \bar{h} \). We denote by \( I(h) \in \mathcal{I}_i \) the information set containing history \( h \).

These imperfect information games allow us to incorporate incomplete information in the standard way in which Nature moves first and determines agents’ types. Due to the nature of the strong dominance properties we study, we do not need to impose any assumptions on agents’ information.

A strategy for a player \( i \) in game \( \Gamma \) is a function \( S_i \) that maps information sets into actions chosen by the agent at each information set.\(^{18}\) When we want to refer to the strategies of different types \( \succ_i \) of agent \( i \), we write \( S_i(\succ_i) \) for the strategy followed by agent \( i \) of type \( \succ_i \); in particular, \( S_i(\succ_i)(I_i) \) denotes the action chosen by agent \( i \) with type \( \succ_i \) at information set \( I_i \in \mathcal{I}_i \). We use \( S_N(\succ_N) = (S_i(\succ_i))_{i \in \mathcal{N}} \) to denote the strategy profile for all of the agents when the type profile is \( \succ_N = (\succ_i)_{i \in \mathcal{N}} \). An extensive-form mechanism, or simply

\(^{17}\) We will see shortly that it is essentially without loss of generality to assume all information sets are singletons, and so will be able to drop the \( I(h) \) notation and identify each information set with the unique sequence of actions (i.e., history) taken to reach it.

\(^{18}\) We consider pure strategies, but the analysis can be easily extended to mixed strategies.
a mechanism, is an extensive-form game $\Gamma$ together with a profile of strategies $S_N$. Two extensive-form mechanisms $(\Gamma, S_N)$ and $(\Gamma', S'_N)$ are equivalent if for every profile of types $\succ_N = (\succ_i)_{i \in N}$, the distribution over outcomes—$(\Gamma, S_N)(\succ_N)$—when agents follow $S_N(\succ_N)$ in $\Gamma$ is the same as when agents follow $S'_N(\succ_N)$ in $\Gamma'$.

3 Example: Obvious Dominance and Millipede Games

How to define the concept of games that are “simple to play”? As an example, we re-examine obvious strategy-proofness, the seminal simplicity concept proposed by Li (2017b). Given a game $\Gamma$, a strategy $S_i$ obviously dominates another strategy $S'_i$ for player $i$ if, starting from any earliest information set $I_i$ at which these two strategies diverge, the worst possible payoff to the agent from playing $S_i$ is at least as good as the best possible payoff from $S'_i$, where the best/worst case outcomes are determined over all possible strategies of other agents $S_{-i}$ and all possible choices of Nature. A profile of strategies $S_N(\cdot) = (S_i(\cdot))_{i \in N}$ is obviously dominant if for every player $i$ and every type $\succ_i$, the strategy $S_i(\succ_i)$ obviously dominates every other strategy $S'_i$. $\Gamma$ is obviously strategy-proof (OSP) if there exists a profile of strategies $S_N(\cdot)$ that is obviously dominant. As Li (2017b) shows, obvious strategy-proofness successfully differentiates between ascending auctions—in which the standard strategy (drop out when the price hits your value) is obviously dominant—and sealed-bid second-price auctions—in which the standard strategy (bid truthfully) is dominant, but is not obviously dominant—even though the two auction formats are payoff-equivalent.

We show that some obviously dominant strategies are not necessarily simple: as we discussed in the introduction, if White has a winning strategy in chess, then this strategy is obviously dominant. Despite this classification, not only we cannot calculate the White’s winning strategy, we do not even know whether it exists. This motivates two natural questions: (1) in what games do all players have obviously dominant strategies? and (2) can we formally define a simplicity standard that better delineates classes of games that are generally understood to be simple?

We first tackle the first of these questions and characterize the entire class of obviously strategy-proof mechanisms in environments without transfers, which are environments

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19 The equivalence concept here is outcome-based, and hence different from the procedural equivalence concept of Kohlberg and Mertens (1986).

20 That is, information set $I_i$ is on the path of play under both $S_i$ and $S'_i$ and both strategies choose the same action at all earlier information sets but choose a different action at $I_i$. Li (2017b) refers to such an information set as an earliest point of departure. Note that for two strategies, there will in general be multiple earliest points of departure.

21 Li (2017b) also provides empirical evidence that people do indeed play the obviously dominant strategy in the ascending version at higher rates than the dominant strategy in the sealed-bid version.
defined by the assumption that $x \succeq_i y$ if and only if both $x$ and $y$ are in the same element of $i$’s equivalence partition. We call the resulting class of games millipedes. The construction of millipede games plays a role in our analysis of the second question because the more demanding simplicity concepts we propose below delineate classes of games that—in environments without transfers—are subsets of the millipede class.

Roughly speaking, a millipede game is a take-or-pass game similar to a centipede game, but with more players and more actions (i.e., “legs”) at each node. Figure 1 shows the extensive form of a millipede game for the special case of object allocation with single-unit demand, where the agents are labeled $i, j, k, \ldots$ and the objects are labeled $w, x, y, \ldots$. At the start of the game, the first mover, agent $i$ has three options: he can take $x$, take $y$, or pass to agent $j$. If he takes an object, he leaves the game and it continues with a new agent. If he passes, then agent $j$ can take $x$, take $z$, or pass back to $i$. If he passes back to $i$, then $i$’s possible choices increase from his previous move (he can now take $z$). The game continues in this manner until all objects have been allocated.

While Figure 1 considers an object allocation environment, we define millipede games more generally for all environments without transfers. Recall that each agent’s preference domain $\mathcal{P}_i$ partitions the outcome space $\mathcal{X}$ into equivalence classes, with each element referred to as a payoff for agent $i$. We say that a payoff $x$ is possible for agent $i$ at history $h$ if there is a strategy profile of all the agents (including choices made by Nature) such that, starting from $h$ and following this strategy profile results in a terminal history $\tilde{h}$ at which agent $i$ obtains payoff $x$. For any history $h$, $\mathcal{P}_i(h)$ denotes the set of payoffs that are possible for $i$ at $h$.\footnote{We provide more formal definitions of the concepts introduced here in the appendix.}
We say agent \( i \) has **clinched** payoff \( x \) at history \( h \) if agent \( i \) receives payoff \( x \) at all terminal histories \( \tilde{h} \supseteq \overline{h} \). If after following action \( a \in A(h) \), an agent receives the same payoff for every terminal \( \tilde{h} \supseteq (h, a) \), we say that \( a \) is a **clinching** action. We denote the set of payoffs that \( i \) can clinch at history \( h \) at which she moves by \( C_i(h) \); that is, \( x \in C_i(h) \) if there is some action \( a \in A(h) \) such that \( i \) receives payoff \( x \) for all terminal \( \tilde{h} \supseteq (h, a) \). At a terminal history \( \tilde{h} \), no agent is called to move and there are no actions; however, for the purposes of constructing millipede games below, it will be useful to define \( C_i(\tilde{h}) = \{ x \} \) for all \( i \), where \( x \) is the payoff associated with the unique outcome that obtains at terminal history \( \tilde{h} \). We further define \( C^C_i(h) = \{ x : x \in C_i(h') \text{ for some } h' \subseteq h \text{ s.t. } i_{h'} = i \} \) to be the set of payoffs that \( i \) can clinch at some subhistory of \( h \), and \( C^C_i(h) = \{ x : x \in C_i(h') \text{ for some } h' \subseteq h \text{ s.t. } i_{h'} = i \} \) to be the set of payoffs that \( i \) can clinch at some strict subhistory of \( h \). (Note that the definition of \( C_i(h) \) implicitly presumes that \( i_h = i \), i.e., \( i \) moves at \( h \); however, \( P_i(h), C^C_i(h) \) and \( C^C_i(h) \) are defined for any \( h \), whether \( i \) moves at \( h \) or not.) If an action \( a \in A(h) \) is not a clinching action, then it is called a **passing** action.

Given these definitions, we define a **millipede game** as a finite extensive-form game of perfect information that satisfies the following properties: Nature either moves once, at the empty history \( h_0 \), or Nature has no moves. At any other history \( h \neq h_0 \), all but at most one action are clinching actions; the remaining action (if there is one) is a passing action.\(^{23}\)

And finally, for all \( i \), all histories \( h \) such that either \( i_h = i \) or \( h \) is a terminal history, and all payoffs \( x \), at least one of the following holds: (a) \( x \in P_i(h) \); or (b) \( x \notin P_i(\tilde{h}) \) for some \( \tilde{h} \in \{ h' : h' \subseteq h \text{ and } i_{h'} = i \} \); or (c) \( x \in C^C_i(\tilde{h}) \); or (d) \( C^C_i(\tilde{h}) \subseteq C_i(h) \).

Conditions (a)-(d) ensure that, if an agent’s top still-possible outcome is not clinchable at some history \( h \), then she is able to guarantee that the payoff she ultimately receives is at least as good as any payoff she could have clinched at \( h \), which is necessary for passing to be obviously dominant. To see this, consider an agent whose top payoff is \( x \), and a history \( h \) such that (a), (b) and (c) fail for \( x \). This means at all prior moves, \( x \) was possible (by “not (b)”), but not clinchable (by “not (c)”), and so obvious dominance requires \( i \) to pass. However, \( x \) has disappeared as a possibility at \( h \) (by “not (a)”), and so we must offer her the opportunity to clinch anything she could have clinched previously (condition (d)), or else she may regret her decision to pass at an earlier node.

Notice that millipede games have a recursive structure: the continuation game that follows any action is also a millipede game. A simple example of a millipede game is a deterministic serial dictatorship in which no agent ever passes and there is only one active

\(^{23}\)There may be several clinching actions associated with the same payoff for \( i \).
agent at each node. A more complex example is sketched in Figure 1.

Our first main result is to characterize the class of OSP games and mechanisms as the class of millipede games with greedy strategies. To define greedy strategies, let $\text{Top}(\succ_i, P_i(h))$ denote the best payoff in the set $P_i(h)$ for an agent of type $\succ_i$. A strategy $S_i(\succ_i)$ is called greedy if, for any $h$ at which $\text{Top}(\succ_i, P_i(h)) \in C_i(h)$, the action $S_i(\succ_i)(h)$ clinches this outcome for the agent; otherwise, the agent passes.

**Theorem 1.** In environments without transfers, every obviously strategy-proof mechanism $(\Gamma, S_N)$ is equivalent to a millipede game with greedy strategies. Every millipede game with greedy strategies is obviously strategy-proof.

This theorem is applicable in many environments. This includes allocation problems in which agents care only about the object(s) they receive, in which case, clinching actions correspond to taking a specified object and leaving the remaining objects to be distributed amongst the remaining agents. Theorem 1 also applies to standard social choice problems in which no agent is indifferent between any two outcomes (e.g., voting), in which case clinching corresponds to determining the final outcome for all agents. In such environments, Theorem 1 implies the following:

**Corollary 1.** If each agent has strict preferences among all outcomes, then every OSP game is equivalent to a game in which either there are only two outcomes that are possible when the first agent moves (and the first mover can either clinch any of them, or can clinch one of them or pass to a second agent, who is presented with an analogous choice, etc.), or the first agent to move can clinch any possible outcome and has no passing action.

The latter case of Corollary 1 is the standard dictatorship, with a possibly restricted set of possible outcomes, while the former case is a generalization that allows for the possibility that at her turn, an agent is a “dictator” over one of the outcomes, but not the other. In particular, this corollary gives an analogue of the Gibbard-Satterthwaite dictatorship result, with no efficiency assumption.

The full proof of Theorem 1 is in the appendix, but here, we provide a brief sketch of the more interesting direction that for any OSP game $\Gamma$, we can find an equivalent millipede game. First, notice that breaking information sets only shrinks the set of possible outcomes, which preserves the min/max obvious dominance inequality, and so every OSP game $\Gamma$ is

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24 An agent is active at history $h$ of a millipede game if the agent moves at $h$, or the agent moved prior to history $h$ and has not yet clinched an outcome.

25 The first more complex example of a millipede game we know of is due to Ashlagi and Gonczarowski (2018). They construct an example of OSP-implementation of deferred acceptance on some restricted preference domains. On these restricted domains, DA reduces to a millipede game (though they do not classify the actions as “passing” or “clinching” actions).
equivalent to a perfect information OSP game $\Gamma'$ in which Nature moves once, as the first mover.\footnote{That every OSP game is equivalent to an OSP game with perfect information was first pointed out in a footnote by Ashlagi and Gonczarowski (2018), which also notes that de-randomizing an OSP game leads to an OSP game. For completeness, the appendix contains the (straightforward) proofs of these statements.} Second, we show that if there are two passing actions $a$ and $a'$ at some history $h$, then there are (by definition) at least two payoffs that are possible for $i$ following each. Obvious dominance then implies that $i$ must have some strategy that can guarantee his first-choice payoff in the continuation game following at least one of $a$ or $a'$, and we can construct an equivalent game in which we replace this action with an equivalent clinching action that allows $i$ to clinch this first-choice payoff already at $h$ by making all such “future choices” today. This procedure can be repeated until there is at most one passing action remaining. Finally, if there remains some $h$ such that agent $i$ cannot clinch her favorite possible payoff at $h$, we must promise $i$ that she will never be strictly worse off by passing (by OSP), which implies that one of the conditions (a)-(d) in definition above must obtain.

Remark 1. As this proof discussion illustrates, we prove a claim stronger than the equivalence of obviously strategy-proof mechanisms and millipedes with greedy strategies. Every obviously strategy-proof game can be transformed into a millipede by four transformations: (i) transforming the imperfect information game into a perfect information one by breaking all information sets; (ii) Li’s pruning, in which the actions no type chooses are removed from the game tree; (iii) having Nature make all its choices at the beginning of the game; and (iv) replacing “future choices” that guarantee a payoff for an agent into a single clinching action.\footnote{The transformations (i) and (iii) are elucidated in Lemma 1 and transformation (ii) in the discussion of pruning directly preceding this lemma. Transformation (iv) is elucidated in Lemma 3.}

Theorem 1 characterizes the entire class of obviously strategy-proof games. We have already mentioned some familiar dictatorship-like games that fit into this class (e.g., Ran-
dom Priority, also known as Random Serial Dictatorship). Another example of a millipede game is sketched in Figure 2. Here, there are 100 agents \( \{i, j, k_1, \ldots, k_{98}\} \) and 100 objects \( \{o_1, o_2, \ldots, o_{100}\} \) to be assigned. The game begins with agent \( i \) being offered the opportunity to clinch \( o_2 \), or pass to \( j \). Agent \( j \) can then either clinch \( o_{99} \), in which case the next mover is \( k_2 \), or pass back to \( i \), and so on. Now, consider the type of agent \( i \) that prefers the objects in the order of their index: \( o_1 \succ_i o_2 \succ_i \cdots \succ_i o_{100} \). At the very first move of the game, \( i \) is offered her second-favorite object, \( o_2 \), even though her top choice, \( o_1 \), is still available. The obviously dominant strategy here requires \( i \) to pass. However, if she passes, she may not be offered the opportunity to clinch her top object(s) for hundreds of moves. Further, when considering all of the possible moves of the other agents, if \( i \) passes, the game has the potential to go off into thousands of different directions, and in many of them, she will never be able to clinch better than \( o_2 \). Thus, while passing is formally obviously dominant, fully comprehending this still requires the ability to reason far into the future of the game and perform lengthy backwards induction.

4 Simple Dominance

The upshot of the previous section is that some OSP mechanisms, such as Random Priority, are indeed quite simple to play; however, the full class of millipede games is much larger, and contains OSP mechanisms that may be quite complex to play. The reason is that OSP relaxes the assumption that agents fully comprehend how the choices of other agents will translate into outcomes, but it still presumes that they understand how their own future actions affect outcomes. Formally, when checking the obvious dominance inequality, the min and the max are taken only over opponents’ strategies, \( S_{-i} \), fixing the agent’s own strategy, \( S_i \). Thus, while OSP guarantees that when taking an action, agents do not have to reason carefully about what their opponents will do, it still may require that they search deep into the game with regard to their future self, and assumes they know all of their own actions in the future, and exactly how those actions affect the set of possible outcomes. (To return to the illuminating example of chess, it presumes that at the start of the game, White knows exactly what she needs to do at any possible future configuration of the board in order to ensure a victory.)

We propose a class of simplicity concepts that relax the assumption that players can analyze and plan their own actions arbitrarily far into the future of the play. The proposed conceptualization offers a way to relax the foresight assumptions embedded not only in obvious dominance but also in other game theoretic concepts. Otherwise, we maintain the approach pioneered by Li (2017b), in particular the assumptions that players cannot fully
analyze the actions of other players but understand the set of possible outcomes following their own actions.\footnote{Our construction allows a relaxation of the first of these assumptions.}

In order to analyze agents who can plan for only part of the game, we need to allow the agent’s perception of the strategic situation, and hence, the planned actions—referred to as a *strategic plan* below, to distinguish from the standard game-theoretic notion of a *strategy*—to vary across nodes at which the agent moves. We first formalize this idea for games of perfect information; the generalization to imperfect information will easily follow.

For each player $i$ and node $h^* \in \mathcal{H}_i$ at which $i$ moves, there is a set of nodes $\mathcal{H}_{i,h^*} \subseteq \{h \in \mathcal{H}_i | h \supseteq h^*\}$ that are perceived as simple from the perspective of node $h^*$.\footnote{The assumption that $\mathcal{H}_{i,h^*} \subseteq \mathcal{H}_i$ is made for simplicity; in its absence we need to endow players with beliefs of what other players will do.} A *strategic plan* for agent $i$ at node $h^*$ is a mapping $S_{i,h^*}$ from $\mathcal{H}_{i,h^*}$ to actions at these nodes.\footnote{We focus on pure strategies; the extension to mixed strategies is straightforward.} Note that a strategic plan does not give an instruction for all continuation nodes at which $i$ may be called to move, but rather only for those nodes in the set $\mathcal{H}_{i,h^*}$. Strategic plan $S_{i,h^*}$ is *simple* at node $h^*$ if the worst possible payoff for $i$ in the continuation game in which $i$ follows $S_{i,h^*}(h)$ at all $h \in \mathcal{H}_{i,h^*}$ is weakly preferred by $i$ to the best possible payoff for $i$ of the continuation game in which $i$ does not play $S_{i,h^*}(h^*)$ at node $h^*$. A set of strategic plans, $(S_{i,h^*})_{h^* \in \mathcal{H}_i}$, one for each node $h^* \in \mathcal{H}_i$ at which $i$ moves, is called a *strategic collection*. A strategic collection $(S_{i,h^*})_{h^* \in \mathcal{H}_i}$ is *simple* if all its strategic plans are simple.

This approach to simplicity takes the collection $(\mathcal{H}_{i,h^*})_{h^* \in \mathcal{H}_i}$ of *simple-node sets* to be a parameter of the definition. The smaller (in an inclusion sense) the set of simple nodes, the stronger is the resulting simplicity requirement. A natural requirement on the collection of simple node sets is that if an agent classifies a node $h \supseteq h_1$ as simple from the perspective of node $h_1$ then the agent continues to classify the node $h$ as simple from the perspective of all nodes $h_2 \supseteq h_1$ such that $h \supset h_2$; we do not impose this requirement but it is satisfied in our main examples to which we turn now.

The generality of our framework allows us to embed important special cases by varying the simple-node sets $(\mathcal{H}_{i,h^*})_{h^* \in \mathcal{H}_i}$. When, for any $h^* \in \mathcal{H}_i$, the set $\mathcal{H}_{i,h^*}$ is the set of all continuation of $h^*$ at which $i$ moves (that is, $i$ can plan all of his future moves), we refer to any resulting simple strategic collection an *obviously dominant strategic collection*. When $\mathcal{H}_{i,h^*} = \{h^*\}$ (that is, $i$ cannot plan any future moves), the resulting simple strategic collections are called *strongly obviously dominant strategic collections*, using the terminology from the 2016 draft of our paper. Another natural instance of the concept is when $\mathcal{H}_{i,h^*} = \{h \in \mathcal{H}_i (h^*) | h^* \subset h \Rightarrow h' \not\in \mathcal{H}_i\}$—that is, when $i$ can plan one move ahead but not more. We refer to simple strategic collections for this set of simple nodes as...
one-step foresight (OSF) dominant strategic collections.

A strategic collection is consistent if $S_{i,h^*}(h) = S_{i,h}(h)$ for all $h \in \mathcal{H}_{i,h^*}$ and all $h^* \in \mathcal{H}_i$. Obviously dominant strategic collections and strongly obviously dominant strategic collections are consistent, while one-step foresight strategic collections need not be consistent. For any consistent strategic collection, we can naturally define the induced (global) strategy as $S_i(h) = S_{i,h}(h)$ for all $h \in \mathcal{H}_i$. It is easy to see that obviously dominant strategic collections induce strategies that are obviously dominant in the sense of Li (2017b), while strongly obviously dominant strategic collections induce strategies that are strongly obviously dominant in the sense we defined in the 2016 draft of our paper. Furthermore, any obviously dominant strategy naturally induces an obviously dominant strategic collection, and the same for strongly obviously dominant strategies.

The extension to imperfect information games is straightforward: we replace nodes $h^*$ and $h$ by information sets $I^*$ and $I$, and replace the relationship $h \supset h^*$ by the relationship that $I$ is a continuation information set of $I^*$ (that is, $I$ a possible information set following $I^*$). The key parameter of the simplicity definition is then the collection $(I_i,I_{\cdot},I_{\cdot})_{I^* \in \mathcal{I}_i}$ of simple information sets, and the simple strategic collections $(S_{i,I^*},I_{\cdot})_{I^* \in \mathcal{I}_i}$ are defined analogously to the discussion above. In the sequel, we focus on perfect information game because of the following.

**Theorem 2.** Let $\Gamma$ be a game of imperfect information, and consider a set of simple information sets $(I_{i,I^*})_{I_{\cdot} \in \mathcal{I}_i}$ and a corresponding simple strategic collection $(S_{i,I^*},I_{\cdot})_{I^* \in \mathcal{I}_i}$. In the perfect information game in which all information sets contain exactly one history from game $\Gamma$, the induced strategic collection $(S_{i,h^*})_{h^* \in \mathcal{H}_i}$ is simple, where each $S_{i,h^*}$ is defined as $S_{i,h^*}(h) = S_{i,I^*}(I)$, where $I$ is a continuation information set of $I^*$ and $h^* \in I^*$ and $h \in I$.

This result obtains from the analogous observation about obvious dominance—first mentioned by Ashlagi and Gonczarowski (2018) and formalized in our Lemma 1—and the first part of the following theorem.

**Theorem 3.** Fix a collection of simple information sets and a strategic collection $(S_{i,I^*},I_{\cdot})_{I^* \in \mathcal{I}_i}$. If the strategic collection is simple, then the induced strategy $S_i(I^*) = S_{i,I^*}(I^*)$ is obviously dominant. Furthermore, if the induced strategy $S_i(I^*) = S_{i,I^*}(I^*)$ is strongly obviously dominant, then the strategic collection is simple.

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31 In the 2016 draft of our paper, we say $S_i$ strongly obviously dominates strategy $S'_i$ in game $\Gamma$ if, starting at any earliest point of departure $h$ between $S_i$ and $S'_i$, the best possible outcome from following $S'_i$ at $h$ is weakly worse than the worst possible outcome from following $S_i$ at $h$, where the best and worst cases are determined by considering any future play by other agents (including Nature) and agent $i$. If a strategy $S_i$ strongly obviously dominates all other $S'_i$, then we say that $S_i$ is strongly obviously dominant.
This result establishes obvious dominance and strong obvious dominance as the extreme points of the class of simple dominance concepts we study. It holds true because the larger (in inclusion sense) are sets of simple information sets the more demanding is the simple dominance requirement.

4.1 Confusion Among Similar Games

We may think of simple strategic collections as providing the right guidance to the player even if the player is confused about the action sets at non-simple nodes. We formalize this idea as follows. For expositional simplicity, we restrict attention to perfect information games. For any game \( \Gamma \) and collection of permutations \( \tilde{\eta} = \{ \eta_h \}_{h \in \mathcal{H}} \) of actions at nodes \( h \in \mathcal{H} \), we construct the relabeled game \( \tilde{\eta}(\Gamma) \) by permuting actions at each node \( h \) by permutation \( \eta_h \); otherwise game \( \tilde{\eta}(\Gamma) \) has the same game tree as \( \Gamma \) and the same payoffs at terminal nodes. For instance, if \( (h^*, a_1, a_2, \ldots, a_k) \) is a terminal history in game \( \Gamma \) then it is a terminal history in game \( \tilde{\eta}(\Gamma) \) and all players payoffs are the same in both games. For a set of simple nodes \( \mathcal{H}_{i,h^*} \), we say that two games \( \Gamma \) and \( \Gamma' \) are indistinguishable from the perspective of node \( h^* \) if there is a collection of permutations \( \tilde{\eta} = \{ \eta_h \}_{h \in \mathcal{H}} \) such that (i) \( \eta_h \) is an identity for all \( h \in \mathcal{H}_{i,h^*} \) and (ii) \( \Gamma' = \tilde{\eta}(\Gamma) \).

This preparation allows us to relate simple strategic collections to the standard notion of weak dominance. We say that a strategy \( S_i \) of player \( i \) is weakly dominant in the continuation game beginning at \( h^* \) if following this strategy leads to weakly better outcomes for \( i \) than any other strategy \( S_i' \), irrespective of the strategies followed by other players.\(^{32}\)

**Theorem 4.** For each game \( \Gamma_i \), agent \( i \), preference ranking \( \succ_i \), and collection of simple histories \( (\mathcal{H}_{i,h^*})_{h^* \in \mathcal{H}_i} \), the strategic plan \( S_{i,h^*} \) is simple from the perspective of \( h^* \in \mathcal{H}_i \) in \( \Gamma \) if and only if in every game \( \Gamma' \) that is indistinguishable from the perspective of node \( h^* \), every strategy \( S_i \) such that \( S_i(h) = S_{i,h^*}(h) \) for all \( h \in \mathcal{H}_{i,h^*} \) is weakly dominant in the continuation game starting at \( h^* \).

The straightforward proof is in the appendix, and—similarly to the proofs of the previous two theorems—it does not rely on our domain richness assumptions.

This theorem tells us the strategic collection \( (S_{i,h^*})_{h^* \in \mathcal{H}_i} \) is simple in \( \Gamma \) if and only if for every \( h^* \in \mathcal{H}_i \) in every game \( \Gamma' \) that is indistinguishable from the perspective of node \( h^* \) with the set of simple nodes \( \mathcal{H}_{i,h^*} \), every strategy \( S_i \) in the continuation game starting at \( h^* \) such that \( S_i(h) = S_{i,h^*}(h) \) for all \( h \in \mathcal{H}_{i,h^*} \) is weakly dominant. When the strategic collection is consistent, we can express this result equivalently in terms of simplicity of the

\(^{32}\)Note that here, \( S_i \) and \( S_i' \) denote full strategies in the standard game-theoretic sense of a complete contingent plan of action.
induced global strategies $S_i(h) = S_{i,h}(h)$. When expressed in this way, this result contains Li’s microfoundation for obvious strategy-proofness.\(^{33}\)

### 4.2 One-Step Foresight

A more demanding simplicity concept than obvious dominance is one-step foresight (OSF) dominance. OSF dominance presumes that agents cannot plan arbitrarily far into the future of the game, but can only plan one move ahead at a time. Formally, OSF dominant strategic collections obtain from simple strategic collections when the simple-node sets $(H_{i,h^*})_{h \in H_i}$ are such that $H_{i,h^*} = \{ h \in H_i(h^*) \mid h^* \subseteq h' \subseteq h \Rightarrow h' \not\subseteq H_i \}$; for shorthand, we refer to any $h \in H_{i,h^*} \setminus \{ h^* \}$ as a **next-history** (or, next-node) at $h^*$. A strategic plan $S_{i,h^*}$ is then OSF-dominant if it is simple when any next-history is viewed as simple from $h^*$ for agent $i$, but no other histories are simple from the perspective of $h^*$.

**Binary allocation with transfers**

One of the main applications of obvious dominance analyzed by Li (2017b) is to binary allocation with transfers, the simplest example of which is the allocation of a single good with transfers. Li (2017b) shows that the canonical ascending (clock) auction is obviously strategy-proof, while the normal-form equivalent second-price sealed-bid auction is not. We strengthen this result and show that ascending auctions satisfy the stronger property of one-step foresight simplicity. This will follow formally as a corollary of Theorem 5, but, given the framework developed in Section 4, this result can be seen immediately, because the following collection of strategic plans is OSF-dominant in an ascending auction: For any $h^*_i$ such that the going price is $p$ is weakly lower than the bidder value $v$: $i$ stays In, with a plan to drop Out at any next-history $h_i \supseteq h^*_i$. For any $h^*_i$ such that $p < v$: $i$ drops Out immediately.\(^{34}\)

**Corollary 2.** *Ascending clock auctions are one-step-foresight simple.*

Li (2017b) goes beyond just ascending clock auctions, and shows that in binary allocation settings with transfers, the class of OSP games is characterized by the class of **personal clock auctions**. As we show next, personal clock auctions are also OSF-simple, and so, surprisingly, any OSP-implementable social choice rule is also implementable in OSF-dominant strategic collections.

\(^{33}\)This result also subsumes the microfoundation for strong obvious strategy-proofness from the 2016-2018 drafts of our paper.

\(^{34}\)Another OSF-simple collection of strategic plans is: For any $h^*_i$ such that the going price is $p$ is strictly lower than the bidder value $v$: Stay In, with plan to drop Out at any next $h_i \supseteq h^*_i$. For any $h^*_i$ such that $p \geq v$: Drop out immediately.
To formally define binary allocation problems with transfers, let $\mathcal{Y} \subseteq 2^N$ be a set of possible allocations and $w \equiv (w_i)_{i \in \mathcal{N}} \in \mathcal{W}^N$ be a profile of transfers for each agent. The set of outcomes is thus $\mathcal{X} = \mathcal{Y} \times \mathcal{W}^N$, and for any outcome $x \in \mathcal{X}$, we write $x = (y, w)$, where $y \in \mathcal{Y}$ and $w \in \mathcal{W}^N$. To be consistent with Li (2017b), in this section, we denote agents types by $\theta_i$, and assume that agents have quasilinear preferences, where $u_i(\theta_i, y, w) = 1_{i \in y} \theta_i + w_i$ denotes agent $i$’s utility from outcome $(y, w)$ when she has type $\theta_i$.\textsuperscript{35} For a game $\Gamma$, define outcome functions $g$ such that $g_y(\bar{h}) \in \mathcal{Y}$ is the allocation at terminal history $\bar{h}$, and $g_{w,i}(\bar{h}) \in \mathcal{W}$ is the transfer to agent $i$ at $\bar{h}$. The following definition of a personal clock auction is adapted from Li (2017b).\textsuperscript{36}

$\Gamma$ is a **personal clock auction** if, for every $i \in \mathcal{N}$, at every earliest history at which $i$ moves $h_i^*$, there exists a fixed transfer $\bar{w}_i \in \mathcal{W}$, a going transfer $\bar{w}_i : \{h_i : h_i \succeq h_i^*\} \to \mathcal{W}$ and a set of “quitting actions” $A_i^q$ such that

- **In Transfer Falls:**
  - For all terminal $\bar{h} \supset h_i^*$, either (i) $i \notin g_y(\bar{h})$ and $g_{w,i}(\bar{h}) = \bar{w}_i$ or (ii) $i \in g_y(\bar{h})$ and $g_{w,i}(\bar{h}) = \min\{\bar{w}_i(h_i) : h_i^* \subseteq h_i \subseteq \bar{h}\}$.
  - If $\bar{h} \supseteq (h, a)$ for some $h \in \mathcal{H}_i$ and $a \in A_i^q$, then $i \notin g_y(\bar{h})$.
  - $A_i^q \cap A(h_i^*) \neq \emptyset$
  - For all $h_i', h_i'' \in \{h_i : h_i \succeq h_i^*\}$:
    - If $h_i' \subset h_i''$, then $\bar{w}_i(h_i') \geq \bar{w}_i(h_i'')$.
    - If $h_i' \subset h_i''$, $\bar{w}_i(h_i') > \bar{w}_i(h_i'')$ and there is no $h_i'''$ such that $h_i' \subset h_i''' \subset h_i''$, then $A_i^q \cap A(h_i'') \neq \emptyset$
    - If $h_i' \subset h_i''$ and $\bar{w}_i(h_i') > \bar{w}_i(h_i'')$, then $|A(h_i') \setminus A_i^q| = 1$
    - If $|A(h_i') \setminus A_i^q| > 1$, then there exists $a \in A(h_i')$ such that, for all $\bar{h} \supset (h_i', a)$, $i \in g_y(\bar{h})$.

- **Out Transfer Falls:**
  - As above, but replace every instance of “$i \in g_y(\bar{h})$” with “$i \notin g_y(\bar{h})$” and vice-versa.

\textsuperscript{35}Note that Li (2017b) allows for a continuum of transfers and types, while we assume everything is finite. Our OSF concept extends straightforwardly to this environment.

\textsuperscript{36}In light of our notion of equivalent mechanisms, the presentation below is a simplification of Definition 15 of Li (2017b): for any personal clock auction that satisfies Definition 15 of Li (2017b), there is an equivalent mechanism that satisfies the definition below.
Intuitively, a personal clock auction is a generalization of classic ascending/descending auctions. In a standard ascending auction for a single good, there is a single price for all agents; at each history, agents choose from one of two actions, either “quit” or “continue”; when an agent quits, she does not win the object (i.e., is “out of the allocation”), and receives a transfer of zero. Personal clock auctions generalize this procedure in several ways: agents may face different prices (“clocks”); at any history, there may be multiple quitting or multiple continuing actions; when an agent quits, her transfer need not be zero; some agents may have clocks that go in different directions (a two-sided clock auction). The key restrictions are that the “clock” for each can only go in one direction (i.e., the transfer either only rises, or it only falls), and, whenever the transfer an agent faces strictly changes, she must be offered an opportunity to quit. But, these restrictions also ensure that there is an OSF-dominant strategic plan at any $h_i$. In particular, we have the following result.

**Theorem 5.** In binary allocation environments with transfers, every OSF game is equivalent to a personal clock auction.

This result follows readily from Li (2017b): since he shows that all OSP games in binary allocation environments can be implemented via personal clock auctions, to prove the theorem, all that needs to be shown is that personal clock auctions are OSF-simple, which, given the discussion above, is straightforward to check. Further, the standard ascending clock auction is a special case of a personal clock auction, and so Corollary 2 above follows as an immediate consequence of Theorem 5.

**Environments without transfers**

In Section 3, we saw that in no-transfer environments, some millipede games, while obviously strategy-proof, could still be quite complex (e.g., Figure 2). Imposing the stronger concept of OSF-simplicity eliminates such complex millipede games, and leaves only games that are monotonic in the following sense: a game $\Gamma$ is monotonic if, for any agent $i$ and any histories $h \subseteq h'$ such that $i$ moves at $h$, $h'$ and does not move at any $h''$ such that $h \subseteq h'' \subseteq h'$, or $h'$ is a terminal history, either (i) $C_i(h) \subseteq C_i(h')$ or (ii) $P_i(h) \setminus C_i(h) \subseteq C_i(h')$. In words, this says that at any next-history for $i$, she is offered to clinch either (i) everything she could have clinched at her previous move or (ii) everything that was possible, but not clinchable at her previous move.

**Theorem 6.** Consider an environment without transfers, and let $\Gamma$ be a millipede game. $\Gamma$ is one-step-foresight simple if and only if it is monotonic.

From the perspective of an agent playing in a game, monotonic games seem particularly simple: each time an agent is called to move, she knows that if she chooses to pass (i.e., not
clinching), at her next move, she will either be able to clinch everything she is offered to clinch currently, or she will be able to clinch her top remaining choice. On the other hand, in a non-monotonic game such as that in Figure 2, an agent’s possible clinching options at future moves may be strictly worse for her for many moves in the future, before eventually being re-offered what she was able to clinch in the past. If agents are unable to plan far ahead in the game tree, it may be difficult to recognize that passing is obviously dominant; in a monotonic game, however, agents only need to be able to plan at most one step at a time to recognize that passing is an optimal strategy.

4.3 Strong Obvious Strategy-Proofness and Price Mechanisms

In light of Theorem 3, the strongest simplicity concept in our class is strong obvious dominance. To remind, a strategy $S_i$ is strongly obviously dominant if, for any other strategy $S'_i$, starting at any earliest point of departure $h$ between $S_i$ and $S'_i$, the worst possible outcome from following $S_i$ is weakly better than the best possible outcome following $S'_i$, where the best and worst cases are taken over all future actions of other agents (including Nature) and all future actions of agent $i$. In the framework of strategic plans/collections, strong obvious dominance obtains for simple strategic collections when the set of simple nodes from the perspective of $h^*$ is $H_{i,h^*} = \{h^*\}$. If a game $\Gamma$ admits a profile of strongly obviously dominant strategies, we say that it is strongly obviously strategy-proof (SOSP). Random Priority is SOSP, but the millipede game depicted in Figure 2 is not. Thus, SOSP mechanisms further delineate the class of games that are simple to play, by eliminating the more complex millipede games that may require significant forward-looking behavior and backward induction.\textsuperscript{37}

Strong obvious strategy-proofness has several appealing features that capture the idea of a game being simple to play. Since SOSP strengthens OSP by looking at the worst/best case outcomes for $i$ over all possible future actions that could be taken by $i$’s opponents and agent $i$ herself, a strongly obviously dominant strategy is one that is weakly better than all alternative strategies even if the agent is concerned that she might tremble in the future or has time-inconsistent preferences. Further, SOSP games can be implemented so that each agent is called to move at most once. We can actually show a stronger result that highlights the simplicity of SOSP games: in any SOSP game, each agent can have at most one history at which her choice of action is payoff-relevant. Formally, we say a history $h$ at which agent $i$ moves is payoff-irrelevant for this agent if $i$ receives the same payoff at all terminal histories.

\textsuperscript{37}Recall also the example of chess discussed in the introduction: if White can force a win, then any winning strategy of White is obviously dominant, yet the strategic choices in chess seem far from obvious. Chess will not admit a strongly obviously dominant strategy.
if \( \bar{h} \supset h \); if \( i \) moves at \( h \) and this history is not payoff-irrelevant, then it is payoff-relevant for \( i \). The definition of SOSP and richness of the preference domain give us the following.

**Theorem 7.** Along each path of an SOSP game that is on the path of the greedy strategies for some type profile, there is at most one payoff-relevant history for each agent.

This result allows us to further conclude that, for a given game path, the unique payoff-relevant history (if it exists) is the first history at which an agent is called to move.\(^{38}\) While an agent might be called to act later in the game, and her choice might influence the continuation game and the payoffs for other agents, it cannot affect her own payoff.

We can refine Theorem 8 further and show that SOSP games are effectively (personalized) posted price games: at the typical payoff-relevant history an agent is offered a menu of substantive outcomes and prices, and selects one of the alternatives from the menu and is never called to move again. More formally, we say that \( \Gamma \) is a **sequential price game** if it is a perfect-information game in which Nature moves first (if at all). The agents then move sequentially, with each agent called to play at most once. The ordering of the agents and the sets of possible outcomes at each history are determined by Nature’s action and the actions taken by earlier agents. As long as there are either at least three distinct payoffs possible for the agent who is called to move or there is exactly one such payoff, the agent can clinch any of the possible payoffs, while at the same time also selecting a message from a pre-determined set of messages.\(^{39}\) When exactly two payoffs are possible for the agent who moves, the agent can be faced with either a choice between them (clinching and picking an accompanying message), or, he might be given a possibility to clinch one of these payoffs (and picking an accompanying message) and passing (with no message). We can also refer to sequential price games as **curated dictatorships**, a term that stresses the connection to dictatorship games in settings without transfers as well as the freedom the designer in setting the menus.

**Theorem 8.** Every strongly obviously strategy-proof mechanism \((\Gamma, S_X)\) is equivalent to a sequential price mechanism with the greedy strategy. Every sequential price mechanism with the greedy strategy is strongly obviously strategy-proof.

Theorem 8 applies to a wide array of environments. For instance, in an object allocation model without transfers, every SOSP mechanism is effectively a sequential (curated) dictatorship in which agents are called sequentially and offered some subsets of objects that they

\(^{38}\)The on-path restriction is not needed if we consider the class of “pruned” game trees in the sense of Li (2017b).

\(^{39}\)As discussed in Section 3, (S)OSP games may present an agent with several different ways to clinch the same payoff; sending a message is a simple way to encode which of the clinching actions the agent takes. Fixing a clinched payoff, each possible message may affect the future of the game, e.g., by determining who the next mover is, though the agent’s own payoff is not affected.
can clinch. They pick their most preferred set from the menu, and leave the game. In a binary allocation setting setting with a single good and transfers, each agent is approached one at a time, and given a take-it-or-leave-it (TIOLI) offer of a price at which she can purchase the good; if an agent refuses, the next agent is approached, and given a (possibly different) TIOLI offer, etc. If there are multiple objects for sale, each agent is offered a menu consisting of several bundles of objects with associated transfers, and selects her most preferred option from the menu. These are only a few examples covered by Theorem 8; the result holds for any environment that satisfies the richness assumption from Section 2.

5 Random Priority

Thus far, all of our results have been about incentives and what makes a game “simple to play”. As an application of our study of simplicity, we show that OSP combined with natural fairness and efficiency axioms characterize the popular Random Priority (RP) mechanism. In Random Priority, first nature selects an ordering of agents, and then each agent moves in turn and chooses the favorite object among those that remain available given previous agents’ choices.

Random Priority succeeds on all three of the above dimensions: it is simple to play, efficient, and fair.\textsuperscript{40} However, this is only a partial explanation of its success, as to now, it has remained unknown whether there exist other such mechanisms, and, if so, what explains the relative popularity of RP over these alternatives.\textsuperscript{41} Theorem 9 provides an answer to this question: not only does Random Priority have good efficiency, fairness, and incentive properties, it is the \textbf{only} mechanism that does so, thus explaining the widespread popularity of Random Priority in practice.

We study object allocation with single-unit demand, a special case of our general framework: there is a set of agents $\mathcal{N}$ and a set of objects, and each agent is to be assigned exactly one of objects. With slight abuse of notation, in this section we take $\mathcal{X}$ to be the set of objects (rather than global outcomes), and thus each agent has strict preferences $\succ_i$ over $\mathcal{X}$. Our efficiency concept is standard Pareto efficiency: an outcome is \textbf{Pareto efficient} when no other outcome is weakly preferred by all participants and strictly preferred by at least one; a mechanism is \textbf{Pareto efficient} if it generates Pareto efficient outcomes for all

\textsuperscript{40}Pareto efficiency and fairness of RP have been recognized at least since Abdulkadiroğlu and Sönmez (1998) (see Bogomolnaia and Moulin (2001) for analysis of more demanding efficiency concepts), while Li (2017b) established OSP of RP. It is easy to see that the standard extensive-form implementation of RP also satisfies all of our more demanding simplicity requirements.

\textsuperscript{41}Bogomolnaia and Moulin (2001) provide a characterization of RP in the special case of $|\mathcal{N}| = 3$, but their result does not extend to larger markets; Liu and Pycia (2011) provide a characterization using asymptotic versions of standard axioms in replica economies as the market size grows to infinity.
Nature’s choices and agents’ types. Our fairness concept is symmetry: a mechanism \((\Gamma, S)\) is **symmetric** if, for any two agents \(i, j \in \mathcal{N}\), the outcome distribution of the mechanism does not change when we transpose the preference rankings of \(i\) and \(j\) and at the same time transpose the objects the two agents obtain. Informally, the outcome of the mechanism would not change if \(i\) played the role of \(j\) and vice versa.\(^{42}\) The symmetry condition fails in a serial dictatorship in which player 1 chooses first among all outcomes and then player 2 chooses among all remaining outcomes: if they have the same most preferred object then 1 obtains this object in the original serial dictatorship but not in the transposed one. Random Priority orders the agents randomly, and in effect the probability agent 1 obtains the preferred object is the same in before and after the transposition.

**Theorem 9.** An obviously strategy-proof mechanism is symmetric and Pareto efficient if and only if it is equivalent to Random Priority.

That RP is obviously strategy-proof was recognized by Li (2017b), and its Pareto efficiency and symmetry is known at least since Abdulkadiroğlu and Sönmez (1998). The converse implication is new. A key step in the proof of the converse implication is our construction of a bijection between permutations of any deterministic Pareto-efficient millipede and permutations of serial dictatorships such that the outcomes of the permuted millipede and permuted serial dictatorship are exactly the same. Applying a permutation of agents \(\sigma : \mathcal{N} \rightarrow \mathcal{N}\) to a serial dictatorship means that we use \(\sigma\) to change the order in which agents make their choices; similarly, applying the permutations \(\sigma\) to a millipede means that agent \(i\) is given the moves of agent \(\sigma (i)\) (see appendix for more formal treatment). The bijection idea was first employed by Abdulkadiroğlu and Sönmez (1998) and then used by e.g. Pathak and Sethuraman (2011) and Carroll (2014). Our construction of the bijection is highly involved and very different from the bijections of the earlier literature. In the construction, we rely on the properties of the millipedes established by us, and on the properties of Pareto efficient OSP mechanisms subsequently obtained by Bade and Gonczarowski (2017). The bijection argument only proves the special case of the theorem restricted to mechanisms that take the form of a uniform randomization over permutations of a deterministic Pareto-efficient millipede, and another key step of the proof is showing that every relevant symmetric mechanism is equivalent to a lottery over such uniform randomizations. We provide details in the appendix.\(^{43}\)

\(^{42}\)We formalize the concept of the role in the appendix. Because any permutation can be decomposed into a composition of transpositions, we can equivalently state the symmetry property as \(\sigma^{-1} \circ (\Gamma, S) \circ \sigma = (\Gamma, S)\) for all permutations \(\sigma : \mathcal{N} \rightarrow \mathcal{N}\).

\(^{43}\)In addition, in Appendix A, we discuss how our proof innovations allow us to strengthen previous analysis that used the bijection method as well as provide several additional related results on strongly obviously strategy proof mechanisms.
6 Concluding Summary

We study the question of what makes a game “simple to play” in a general class of environments with and without transfers. We introduce a general class of simplicity concepts that vary the foresight abilities required of agents in extensive-form imperfect-information games, and use it to provide characterizations of simple mechanisms.

Li’s (2017) obvious strategy-proofness is the weakest simplicity concept included in our class. Focusing on settings without transfers, we characterize all mechanisms in this class as clinch-or-pass games we call millipede games. Some millipede games are indeed simple and widely-used, though others may be complex, requiring significant foresight on the part of the agents, and are rarely observed.

Another natural concept in our class is one-step foresight. It weakens the foresight abilities assumed of the agents and eliminates these complex millipede games, leaving monotonic games as the only simple games, a class which includes ascending auctions. Building on Li’s characterization of personal clock auctions we show that in the binary environments he studied, every obviously strategy-proof mechanism can be implemented in a one-step-foresight simple way.

The strongest simplicity concept we study is strong obvious strategy-proofness. Strongly obviously strategy-proof mechanisms eliminate the need for significant foresight and backwards induction: we prove that in these mechanisms each agent is asked to make at most one payoff-relevant choice. While a constraint on the choice of mechanisms, strong obvious strategy-proofness is still weak enough to allow positive results, and indeed admits mechanisms that are seen extensively in practice: we show that SOSP mechanisms are equivalent to personalized price mechanisms. Posted prices are among the most popular sale mechanisms. Even on eBay, which began as an auction website, Einav et al. (2018) document a dramatic shift in the 2000s from auctions to posted prices as the predominant selling mechanism on the platform. Our work provides a reason why posted prices are so widely used in practice.\footnote{For an earlier microfoundation of posted prices, see Hagerty and Rogerson (1987). Armstrong (1996) shows that posted prices (combined with bundling) can achieve good revenues (for other analyses of revenues of posted price mechanisms see also e.g. Chawla et al. (2010) and Feldman et al. (2014)). We also refer to personalized-price mechanisms as curated dictatorships because in the special case of our setting in which there are no transfers personalized-price mechanisms resemble sequential dictatorships.}

As a final application we prove a natural characterization of the popular mechanism known as Random Priority by showing that it is the unique mechanism that is obviously strategy-proof, Pareto efficient, and satisfies equal treatment of equals. In other words, our results show that Random Priority is the unique mechanism satisfying these desirable incentive, efficiency, and fairness properties, providing an explanation for its widespread use.
A Additional Results

A.1 Strengthening the Results on Top Trading Cycles

The insight behind Lemma 6 in the proof of Theorem 9 allows us to broaden the applicability of the many results on equivalences of symmetrizations of mechanisms in Pápai (2000) Hierarchical Exchange class and its extension by Pycia and Ünver (2011). Abdulkadiroğlu and Sönmez (1998), Pathak and Sethuraman (2011), Carroll (2014), and Pycia (2016) showed that symmetrizations over such mechanisms are equivalent to Random Priority and the same argument as in Lemma 6 allows to substantially weaken their assumptions and replace it with symmetry. Similarly we can weaken the symmetrization assumption in the main result of Lee and Sethuraman (2011).\footnote{A similar weakening of symmetry assumption in Che and Kojima (2010) is implied by Liu and Pycia (2011), whose proof approach however is very different from ours (cf. also Pycia (2011)).}

A.2 Equal Treatment of Equals

In Section 5 we study symmetric mechanisms. Here we complement Theorem 9 by a similar result in which we strengthen OSP to SOSP and relax symmetry to a weaker assumption imposed on marginal distributions only: a mechanism satisfies equal treatment of equals if, whenever two agents have the same type, they receive the same distribution over payoffs.

**Theorem 10.** Every mechanism that is strongly obviously strategy-proof, Pareto efficient, and satisfies equal treatment of equals is equivalent to Random Priority with greedy strategies. Furthermore, Random Priority with greedy strategies is strongly obviously strategy-proof, Pareto efficient, and satisfies equal treatment of equals.

The proof of this result, presented in Appendix B, does not rely on the bijection approach we employed in Theorem 9; rather, it builds on our characterization of SOSP mechanisms that are Pareto efficient.

A.3 Efficient Strongly Obviously Strategy-Proof Mechanisms

An almost-sequential dictatorship is a perfect-information game in which Nature moves first and then the agents move in turn, with each agent moving at most once. At his move, an agent picks an object and sends a message. As long as there are at least three objects unallocated, the moving agent can choose from all still-available objects. When there are two objects remaining, an agent can be faced with either a choice between them, or, he might be
given a choice between one of these objects for sure or giving the next agent an opportunity to allocate the remaining two objects among the two of them.\footnote{Pycia and Ünver (2016) use the same name for deterministic mechanisms without messages that belong to the class we study; they show that these are exactly the deterministic mechanisms which are strategy-proof and Arrovian efficient with respect to a complete social welfare function. We use the name their introduced because our class is a natural extension of theirs.}

The difference between a serial dictatorship and a sequential dictatorship is that in a serial dictatorship, the ordering of the agents is fixed after Nature’s move, while in a sequential dictatorship, the next agent to move may depend on the choices of the earlier agents. The reason we call the mechanism described above an “almost” sequential dictatorship is because it works exactly as a sequential dictatorship if there are three or more objects remaining; however, a small modification is allowed when only two objects remain.

**Theorem 11.** Every strongly obviously strategy-proof and Pareto efficient mechanism \((\Gamma, S_N)\) is equivalent to an almost-sequential dictatorship with greedy strategies. Every almost-sequential dictatorship with greedy strategies is strongly obviously strategy-proof and Pareto efficient.

That almost-sequential dictatorships are SOSP and efficient is immediate. The proof of the other direction follows easily from Theorem 8; we provide details in Appendix B.\footnote{The natural analogue of Theorem 11 remains valid beyond object allocation, for all rich preference domains, with no substantive changes in the proof.}

## B Proofs

### Preliminary Definitions

Before proceeding with the main proofs, we first define the concepts of possible, guaranteeable, and clinchable outcomes/actions more formally. Fix a game \(\Gamma\). Let \(S = (S_i)_{i \in N}\) denote a strategy profile for the agents. Let \(\omega := (\omega(h))_{h \in H}\) denote one particular realization of Nature’s moves through the game, where \(\omega(h) \in A(h)\) is the action taken by Nature at a history \(h\) at which Nature is called to move. Define \(z(h, S, \omega) \in X\) as the unique final outcome that obtains at the terminal history \(\hat{h}\) that is reached when play starts at some \(h\) and proceeds according to \((S, \omega)\).

We first discuss the distinction between types of payoffs (possible vs. guaranteeable) and then the distinction between types of actions (clinching actions vs. passing actions). Recall that agents may be indifferent between several outcomes. For any outcome \(x \in X\), let \([x]_i = \{y \in X : y \sim_i x\}\) denote the \(x\)-indifference class of agent \(i\), and define

\[
X_i(h, S_i) = \{[x]_i : z(h, (S_i, S_{-i}), \omega) \in [x]_i \text{ for some } (S_{-i}, \omega)\}
\]
to be the possible indifference classes that may obtain for agent $i$ starting at history $h$ if she follows strategy $S_i$. If there exists some $S_i$ such that $[x]_i \in X_i(h, S_i)$, then we then we say that $[x]_i$ is possible for $i$ at $h$. If, further, there exists some $S_i$ such that $X_i(h, S_i) = \{[x]_i\}$, then we say $[x]_i$ is guaranteeable for $i$ at $h$. Let

$$P_i(h) = \{[x]_i : \exists S_i \text{ s.t. } [x]_i \in X_i(h, S_i)\}$$

$$G_i(h) = \{[x]_i : \exists S_i \text{ s.t. } X_i(h, S_i) = \{[x]_i\}\}$$

be the sets of indifference classes that are possible and guaranteeable at $h$, respectively.\(^{48}\) Note that $G_i(h) \subseteq P_i(h)$, and the set $P_i(h) \setminus G_i(h)$ is the set of indifference classes that are possible at $h$, but are not guaranteeable at $h$.

Remark. In all of the proofs below, we will generally drop the bracket notation $[x]_i$ and, when there is no confusion, simply refer to the “payoff $x$”. Statements such as “$x$ is a possible payoff at $h$” or “$x \in P_i(h)$” are understood as “some outcome in the indifference class $[x]_i$ is possible at $h$” (the same applies to clinchable indifference classes/payoffs, defined next).

Last, we define a distinction between two kinds of actions: clinching actions and passing actions. Let $i_h = i$ be the agent who is to act at a history $h$. Using our notational convention that $(h, a)$ denotes the history obtained by starting at $h$ and following action $a$, the set $P_i((h, a))$ is the set of payoffs that are possible for $i$ if she takes action $a$ at $h$. If $P_i((h, a)) = \{[x]_i\}$, then we say that action $a \in A(h)$ clinches payoff $x$ for $i$. If an action $a$ clinches $x$ for $i$, we call $a$ a clinching action. Note that there can be multiple actions in $A(h)$ that clinch the same payoff $x$ for $i$. Any action of an agent that is not a clinching action is called a passing action. Let $C_i(h)$ denote the set of payoffs that are clinchable for $i$ at $h$; that is,

$$C_i(h) = \{[x]_i : \exists a \in A(h) \text{ s.t. } P_i((h, a)) = \{[x]_i\}\}.$$ 

Note that this definition of $C_i(h)$ presumes that agent $i$ is called to play at history $h$. If $\tilde{h}$ is a terminal history, then no agent is called to play and there are no actions. However, it will be useful in what follows to define $C_i(\tilde{h}) = \{[x]_i\}$ for all $i$, where $x$ is the unique outcome associated with the terminal history $\tilde{h}$.

We also remind the reader of two additional pieces of notation that were introduced in Section 3:

$$\overline{C_i(h)} = \{[x]_i : [x]_i \in C_i(h') \text{ for some } h' \subseteq h \text{ s.t. } i_{h'} = i\}$$

\(^{48}\)Note that $P_i(h)$ and $G_i(h)$ are well-defined even if $i_h \neq i$, i.e., even if $i$ is not the agent who moves at $h$. 

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\[ C^C_i(h) = \{ [x]_i : [x]_i \in C_i(h') \text{ for some } h' \subset h \text{ s.t. } i_{h'} = i \}. \]

In words, \( C^C_i(h) \) is the set of payoffs that \( i \) can clinch at some subhistory of \( h \), and \( C^C_i(h) \) is the set of payoffs that \( i \) can clinch at some \textit{strict} subhistory of \( h \). Note that the definition of \( C_i(h) \) implicitly presumes that \( i_h = i \), i.e., \( i \) moves at \( h \); however, \( P_i(h), C^C_i(h) \) and \( C^C_i(h) \) are defined for any \( h \), whether \( i \) is the agent who moves at \( h \) or not.

\section{Proof of Theorem 1}

We break Theorem 1 into two propositions. We start by proving that millipede games are OSP (Proposition 1), and then prove that every OSP game is equivalent to a millipede game (Proposition 2).

\textbf{Proposition 1.} Millipede games with greedy strategies are obviously strategy-proof.

\textit{Proof.} Let \( \Gamma \) be a millipede game. Recall that the \textit{greedy strategy} for any agent \( i \) is defined as follows: for any history \( h \) at which \( i \) moves, if \( i \) can clinch her top payoff in \( P_i(h) \), then \( S_i(\succ_i)(h) \) instructs \( i \) to follow an action that clinches this payoff; otherwise, \( i \) passes at \( h \).

We now show that it is obviously dominant for all agents to follow a greedy strategy. Consider some profile of greedy strategies \((S_i(\cdot))_{i \in \mathcal{N}}\). For any subset of outcomes \( X' \subset X \), define \( \text{Top}(\succ_i, X') \) as the best possible payoff in the set \( X' \) according to preferences \( \succ_i \), i.e., \( x \in \text{Top}(\succ_i, X') \) if and only if \( x \succ_i y \) for all \( y \in X' \) (note that we use our standard convention whereby a payoff \( x \) represents the entire indifference class to which \( x \) belongs, and so \( \text{Top}(\succ_i, X') \) is a singleton). Then, \( \text{Top}(\succ_i, P_i(h)) \) denotes \( i \)'s top payoff among all payoffs that are possible at history \( h \), and \( \text{Top}(\succ_i, C_i(h)) \) denotes \( i \)'s top payoff among all of his clinchable payoffs at \( h \). It is clear that if \( \text{Top}(\succ_i, C_i(h)) = \text{Top}(\succ_i, P_i(h)) \), then the greedy action of clinching the top payoff is obviously dominant at \( h \). What remains to be shown is if \( \text{Top}(\succ_i, C_i(h)) \neq \text{Top}(\succ_i, P_i(h)) \), then passing is obviously dominant at \( h \).

Assume that there exists a history \( h \) that is on the path of play for type \( \succ_i \) when she follows the greedy strategy and \( \text{Top}(\succ_i, C_i(h)) \neq \text{Top}(\succ_i, P_i(h)) \), yet passing is not obviously dominant at \( h \); further, let \( h \) be any earliest such history for which this is true. To shorten notation, let \( x_P(h) = \text{Top}(\succ_i, P_i(h)) \), \( x_C(h) = \text{Top}(\succ_i, C_i(h)) \), and let \( x_W(h) \) be the worst possible payoff from passing (and following \( S_i(\succ_i) \) in the future). Since passing is not obviously dominant, it must be that \( x_W(h) \not\succ_i x_C(h) \).

\footnote{There may be multiple ways for \( i \) to clinch the same payoff \( x \) at \( h \), and further, \( x \) may in principle still be possible/guaranteeable if \( i \) passes at \( h \). Our goal is simply to prove the existence of at least one obviously dominant strategy for \( i \).}
First, note that $x_W(h) \geq_i x_W(h')$ for all $h' \subsetneq h$ such that $i_{h'} = i$. Since passing is obviously dominant at all such $h'$, we have $x_W(h') \geq_i x_C(h')$, and together, these imply that $x_W(h) \geq_i x_C(h')$ for all such $h'$. At $h$, since passing is not obviously dominant, we have $x_C(h) \succ_i x_W(h)$, and further, there must be some $x' \in P_i(h) \setminus G_i(h)$ such that $x' \succ_i x_C(h) \succ_i x_W(h)$.

The above implies that $x' \succ_i x_C(h) \succ_i x_C(h')$ for all $h' \subsetneq h$ such that $i_{h'} = i$. Let $X_0 = \{x' : x' \subset P_i(h) \text{ and } x' \succ_i x_C(h')\}$. In words, $X_0$ is a set of payoffs that are possible at all $h' \subset h$, and are strictly better than anything that was clinchable at any $h' \subset h$ (and therefore have never been clinchable themselves). Order the elements in $X_0$ according to $\succ_i$, and wlog, let $x_1 \succ_i x_2 \succ_i \cdots \succ_i x_M$.

**Definition.** Let $h'$ be a history where either agent $i$ is called to move or $h'$ is a terminal history. We say that payoff $x$ **becomes impossible** for $i$ at $h'$ if: (i) $x \in P_i(h'')$ for all $h'' \subsetneq h'$ such that $i_{h''} = i$ and (ii) $x \notin P_i(h')$.

In other words, the history $h'$ at which a payoff becomes impossible is the earliest history at which $i$ moves and where $x$ “disappears” as a possible outcome for her.

Consider a path of play starting from $h$ and ending in a terminal history $\tilde{h}$ at which type $\succ_i$ of agent $i$ receives his worst case payoff $x_W(h)$. For every $x_m \in X_0$, let $h_m$ denote the history on this path at which $x_m$ becomes impossible for $i$. Note that because $i$ is ultimately receiving payoff $x_W(h)$, such a history $h_m$ exists for all $x_m \in X_0$. Let $\hat{h} = \max\{h_1, h_2, \ldots, h_M\}$ (ordered by $\subset$); in words, $\hat{h}$ is the earliest history at which everything in $X_0$ is no longer possible. Further, let $\hat{h}_{-m} = \max\{h_1, \ldots, h_{m-1}\}$, i.e., $\hat{h}_{-m}$ is the earliest history at which all payoffs strictly preferred to $x_m$ are no longer possible.

**Claim 1.** For all $x_m \in X_0$ and all $h' \subset \hat{h}$, we have $x_m \notin C_i(h')$.

**Proof.** First, note that $x_m \notin C_i(h')$ for any $h' \subset \hat{h}$ by construction. We will show that $x_m \notin C_i(h')$ at any $\tilde{h} \supset h' \supset h$ as well. Start by considering $m = 1$, and assume $x_1 \in C_i(h')$ for some $\tilde{h} \supset h' \supset h$. By definition, $x_1 = \text{Top}(\succ_i, P_i(h))$; since $h' \supset h$ implies that $P_i(h') \subset P_i(h)$, we have that $x_1 = \text{Top}(\succ_i, P_i(h'))$ as well. Since $x_1 \in C_i(h')$ by supposition, greedy strategies direct $i$ to clinch $x_1$, which contradicts that she receives $x_W(h)$.

Now, consider an arbitrary $m$, and assume that for all $m' = 1, \ldots, m - 1$, payoff $x_{m'}$ is not clinchable at any $h' \subset \hat{h}$, but $x_m$ is clinchable at some $h' \subset \hat{h}$. Let $x_{m'}$ be (a) payoff that becomes impossible at $\hat{h}_{-m}$ and is such that $x_{m'} \succ_i x_m$. There are two cases:

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50At least one such $x'$ exists by the assumption that $\text{Top}(\succ_i, C_i(h)) \neq \text{Top}(\succ_i, P_i(h))$, though there in general may be multiple such $x'$.

51It is possible that $h_m$ is a terminal history.

52Recall that for terminal histories $h$, we define $C_i(h) = \{x\}$, where $x$ is the unique payoff associated with the terminal history. Thus, if $h'$ is a terminal history, then $i$ receives payoff $x_1$, which also contradicts that she receives payoff $x_W(h)$. 

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Case (i): \( h' \subset \hat{h}_m \). This is the case where \( x_m \) is clinchable while there is some strictly preferred payoff \( x_{m'} \succ_i x_m \) that is still possible. Note that at history \( \hat{h}_m \), neither (a) nor (b) in the definition of a millipede game hold for \( x_{m'} \). Further, by the inductive hypothesis, \( x_{m'} \) is nowhere clinchable, and so (c) does not hold either. This implies that (d) must hold, and so \( x_m \) must be clinchable at \( \hat{h}_m \). Then, since all preferred payoffs are no longer possible at \( \hat{h}_m \), \( x_m \) is the best possible payoff remaining, and is clinchable. Therefore, greedy strategies instruct agent \( i \) to clinch \( x_m \), which contradicts that she receives \( x_W(h) \).

Case (ii): \( h' \supseteq \hat{h}_m \). In this case, \( x_m \) becomes clinchable after all strictly preferred payoffs are no longer possible. Thus, again, greedy strategies instruct \( i \) to clinch \( x_m \), which contradicts that she is receiving \( x_W(h) \).

To finish the proof, again let \( \hat{h} = \max\{h_1, h_2, \ldots, h_M\} \) and let \( \hat{x} \) be a payoff that becomes impossible at \( \hat{h} \). The claim shows that \( \hat{x} \) is not clinchable at any \( h' \subset \hat{h} \). The preceding two statements imply that conditions (a)-(c) in the definition of a millipede game do not hold at \( \hat{h} \), and so condition (d) must hold, i.e., \( x_C(h) \in C_i(\hat{h}) \). Since \( x_C(h) \) is the best possible remaining payoff at \( \hat{h} \), greedy strategies direct \( i \) to clinch \( x_C(h) \), which contradicts that she receives \( x_W(h) \).

We now prove the second part of the theorem, restated below as Proposition 2. To do so, we first need to introduce the pruning principle of Li (2017b), which will simplify some of the arguments. Given a game \( \Gamma \) and strategy profile \((S_i(\succ_i))_{i \in N}\), the pruning of \( \Gamma \) with respect to \((S_i(\succ_i))_{i \in N}\) is a game \( \Gamma' \) that is defined by starting with \( \Gamma \) and deleting all histories of \( \Gamma \) that are never reached for any type profile. Then, the pruning principle says that if \((S_i(\succ_i))_{i \in N}\) is obviously dominant for \( \Gamma \), the restriction of \((S_i(\succ_i))_{i \in N}\) to \( \Gamma' \) is obviously dominant for \( \Gamma' \), and both games result in the same outcome. Thus, for any OSP mechanism, we can find an equivalent OSP pruned mechanism. When proving this proposition, we assume that all OSP games have been pruned with respect to the equilibrium strategy profile. Note also that we actually prove a slightly stronger statement, which is that every OSP game is equivalent to a millipede game that satisfies the following additional property: for all \( i \), all \( h \) at which \( i \) moves, and all \( x \in G_i(h) \), there exists an action \( a_x \in A(h) \) that clinches \( x \).

**Proposition 2.** Every obviously strategy-proof mechanism \((\Gamma, S_N)\) is equivalent to a millipede game with the greedy strategy.

**Proof.** The proof of this proposition is broken down into several lemmas.

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53If \( \hat{h} \) is a terminal history, then we make an argument analogous to footnote 52 to reach the same contradiction.

54See Lemma 3 below.
Lemma 1. Every OSP game is equivalent to an OSP game with perfect information in which Nature moves at most once, as the first mover.

Proof. Ashlagi and Gonczarowski (2018) briefly mention this result in a footnote; here, we provide the straightforward proof for completeness. We first show that every OSP game is equivalent to an OSP game with perfect information. Denote by $A(I)$ the set of actions available at information set $I$ to the agent who moves at $I$. Take an obviously strategy-proof game and consider its perfect-information counterpart $\Gamma'$, that is the perfect information game at which at every history $h$ in $\Gamma$ the moving agent’s information set is $\{h\}$ in $\Gamma'$, the available actions are $A(I)$, and the outcomes in $\Gamma'$ following any terminal history are the same as in $\Gamma$. Notice that the support of possible outcomes at any history $h$ in $\Gamma'$ is a subset of the support of possible outcomes at $I(h)$ in $\Gamma$. Thus, the worst-case outcome from any action (weakly) increases in $\Gamma'$, while the best-case outcome (weakly) decreases. Thus, if there is an obviously dominant strategy in $\Gamma$, following the analogous strategy in $\Gamma'$ continues to be obviously dominant. Hence, $\Gamma'$ is obviously strategy-proof and equivalent to $\Gamma$.

We now show that every OSP game is equivalent to a perfect-information OSP game in which Nature moves once, as the first mover. Consider a game $\Gamma$, which, by the previous paragraph, we can assume has perfect information. Let $\mathcal{H}_{\text{nature}}$ be the set of histories $h$ at which Nature moves in $\Gamma$. Consider a modified game $\Gamma'$ in which at the empty history Nature chooses actions from $\times_{h \in \mathcal{H}_{\text{nature}}} A(h)$. After each of Nature’s initial moves, we replicate the original game, except at each history $h$ at which Nature is called to play, we delete Nature’s move and continue with the subgame corresponding to the action Nature chose from $A(h)$ at $\emptyset$. Again, note that for any agent $i$ and history $h$ at which $i$ is called to act, the support of possible outcomes at $h$ in $\Gamma'$ is a subset of the support of possible outcomes at the corresponding history in $\Gamma$ (where the corresponding histories are defined by mapping the $A(h)$ component of the action taken at $\emptyset$ by Nature in $\Gamma'$ as an action made by Nature at $h$ in game $\Gamma$). Using reasoning similar to the previous paragraph, we conclude that $\Gamma'$ is obviously strategy-proof, and $\Gamma$ and $\Gamma'$ are equivalent.

Lemma 2. Let $\Gamma$ be an obviously strategy-proof game of perfect information that is pruned with respect to the obviously dominant strategy profile $(S_i(\succ_i))_{i \in N}$. Consider a history $h$ where agent $i_h = i$ is called to move. There is at most one action $a^* \in A(h)$ such that $P_i((h, a^*)) \not\subseteq G_i(h)$.

Proof. For any history $h$, let $PnG_i(h) = P_i(h) \setminus G_i(h)$ (where “PnG” is shorthand for "possible but not guaranteeable"). Now, consider any $h$ at which $i$ moves, and assume that at $h$, there are (at least) two such actions $a_1^*, a_2^* \in A(h)$ as in the statement. We first claim that $PnG_i(h) \cap P_i(h_1^*) \cap P_i(h_2^*) = \emptyset$, where $h_1^* = (h, a_1^*)$ and $h_2^* = (h, a_2^*)$. Indeed, if not,
then let $x$ be a payoff in this set. By pruning, some type $\succ_i$ is following some strategy such that $S_i(\succ_i)(h) = a_i^*$ that results in a payoff of $x$ at some terminal history $\tilde{h} \supset (h, a_i^*)$. Note that $\text{Top}(\succ_i, P_i(h)) \neq x$, because otherwise $a_i^*$ would not be obviously dominant for this type (since $x \notin G_i(h)$ and $x \in P_i(h^*_i)$). Thus, let $\text{Top}(\succ_i, P_i(h)) = y$. Note that $y \notin G_i(h)$ (or else it would not be obviously dominant for type $\succ_i$ to play a strategy such that $x$ is a possible payoff), and it is without loss of generality to assume that $\text{Top}(\succ_i, P_i(h) \setminus \{y\}) = x$. Further, note that $y \in P_i(h^*_1)$ and $y \notin P_i(h^*_2)$. To see the former, note that if $y \notin P_i(h^*_1)$, then $a_i^*$ is not obviously dominant for type $\succ_i$, which contradicts that $S_i(\succ_i)(h) = a_i^*$; given the former, if $y \in P_i(h^*_2)$, then once again $a_i^*$ would not be obviously dominant for type $\succ_i$. Now, again by pruning, there must be some type $\succ_i'$ such that $S_i(\succ_i')(h) = a_i^*$ that results in payoff $x$ at some terminal history $\tilde{h} \supset (h, a_i^*)$. By similar reasoning as previously, $\text{Top}(\succ_i', P_i(h)) \neq x$, and so $\text{Top}(\succ_i', P_i(h)) = z$ for some $z \in P_i(h^*_2)$. Since $y \notin P_i(h^*_2)$, we have $z \neq y$, and we can as above conclude that $z \notin G_i(h)$. Similarly to footnote 55, it is without loss of generality to consider the type $\succ_i'$ that ranks $y$ immediately after $z$. Note that, for this type, no action $a \neq a_i^*$ can obviously dominate $a_i^*$ (since $z \notin G_i(h)$). Further, $a_i^*$ itself is not obviously dominant for this type, since the worst case from $a_i^*$ is strictly worse than $y$, while $y \in P_i(h^*_1)$. Therefore, this type has no obviously dominant action at $h$, which is a contradiction.

Thus, $\text{PnG}_i(h) \cap P_i(h^*_1) \cap P_i(h^*_2) = \emptyset$, which means there must be distinct $x, y$ such that (i) $x, y \in \text{PnG}_i(h)$ (ii) $x \in P_i(h^*_1)$ but $x \notin P_i(h^*_2)$ and (iii) $y \in P_i(h^*_2)$ but $y \notin P_i(h^*_1)$. If there is a type that reaches $h$ such that $\text{Top}(\succ_i, P_i(h)) = x$, then there is also type such that reaches $h$ such that $y$ is ranked immediately after $x$; however, this type would have no obviously dominant action at $h$. The same applies for any type such that $\text{Top}(\succ_i, P_i(h)) = y$. Thus, for all types that reach $h$, it must be that $\text{Top}(\succ_i, P_i(h)) \neq x, y$; further, by pruning, some such type is playing a strategy such that $S_i(\succ_i)(h) = a_i^*$ and $x$ is a possible payoff. Let $\text{Top}(\succ_i, P_i(h)) = z$ for this type. The fact that $S_i(\succ_i)(h) = a_i^*$ implies that $z \in P_i(h^*_1)$ and $z \notin G_i(h)$ (if either were false, then it would not obviously dominant for this type to play a strategy such that $S_i(\succ_i)(h) = a_i^*$ and $x$ is a possible payoff); in other words, $z \in \text{PnG}_i(h)$, and $z \in P_i(h^*_1)$. Since we just showed that $\text{PnG}_i(h) \cap P_i(h^*_1) \cap P_i(h^*_2) = \emptyset$, we have $z \notin P_i(h^*_2)$. Finally, consider a type $\succ_i$ such that $\text{Top}(\succ_i, P_i(h)) = z$ and $\text{Top}(\succ_i, P_i(h) \setminus \{z\}) = y$, and note that this type has no obviously dominant action at $h$.\footnote{Since $h$ is on path for some type such that $y \succ_i x$, it is also on path for the type $\succ_i'$ that is the same as $\succ_i$, except that $\succ_i'$ promotes payoff $x$ to be immediately after $y$.}

Clinching actions are those for which $i$’s payoff is completely determined after following

\footnote{Again, such a type reaches $h$ following footnote 55. Since $z \notin G_i(h)$ and $z \in P_i(h^*_1)$, no action $a \neq a_i^*$ can obviously dominate $a_i^*$. However, the worst case from $a_i^*$ is strictly worse than $y$ (since $z \notin G_i(h)$ and $y \notin P_i(h^*_1)$), while $y \in P_i(h^*_2)$, and so $a_i^*$ itself is also not obviously dominant.}
the action. Lemma 2 shows that if a game is OSP, then at every history, for all actions $a$ with the exception of possibly one special action $a^*$, all payoffs that are possible following $a$ are also guaranteeable at $h$; note, however, it does not say that all actions but at most one are clinching actions. Indeed, it leaves open the possibility that there are several actions that can ultimately lead to multiple final payoffs for $i$, which can happen when different payoffs are guaranteeable for $i$ by following different strategies in the future of the game. The next lemma shows that if this is the case, we can always construct an equivalent OSP game such that all actions except for possibly one are clinching actions.

**Lemma 3.** Let $\Gamma$ be an OSP game of perfect information that is pruned with respect to the obviously dominant strategy profile $(S_i(\succ_i))_{i \in N}$. There exists an equivalent OSP game $\Gamma'$ with perfect information such that the following hold at each $h$ (where $i$ is the agent called to move at $h$):

(i) At least $|A(h)| - 1$ actions at $h$ are clinching actions

(ii) For every payoff $x \in G_i(h)$, there exists an action $a_x \in A(h)$ that clinches $x$ for $i$ and $i_{h'} \neq i$ for all $h' \supset (h, a_x)$.

(iii) If $P_i(h) = G_i(h)$, then all actions in $A(h)$ are clinching actions and $i_{h'} \neq i$ for any $h' \supset h$.

*Proof.* Consider some history $h$ of game $\Gamma$ at which the mover is $i(h) = i$. By Lemma 2, all but at most one action (denoted $a^*$) in $A(h)$ satisfy $P_i((h, a)) \subseteq G_i(h)$; this means that any obviously dominant strategy for type $\succ_i$, that does not choose $a^*$ guarantees the best possible outcome in $P_i(h)$ for type $\succ_i$. Define the set $S_i(h) = \{S_i : S_i(h) \neq a^*$ and $|X(h, S_i)| = 1\}$, and notice that each $S_i \in S_i(h)$ guarantees a unique payoff for $i$ if she plays strategy $S_i$ starting from history $h$, no matter what the other agents do.

We create a new game $\Gamma'$ that is the same as $\Gamma$, except we replace the subgame starting from history $h$ with a new subgame defined as follows. If there is an action $a^*$ such that $P_i((h, a^*)) \not\subseteq G_i(h)$ in the original game (of which there can be at most one), then there is an analogous action $a^*$ in the new game, and the subgame following $a^*$ is exactly the same as in the original game $\Gamma$. Additionally, there are $M = |S_i(h)|$ other actions at $h$, denoted $a_1, \ldots, a_M$. Each $a_m$ corresponds to one strategy $S_i^m \in S_i(h)$, and following each $a_m$, we replicate the original game, except that at any future history $h' \supset h$ at which $i$ is called on to act, all actions (and their subgames) are deleted and replaced with the subgame starting from the history $(h', a')$, where $a' = S_i^m(h')$ is the action that $i$ would have played at $h'$ in the original game had she followed strategy $S_i^m(\cdot)$. In other words, if $i$’s strategy was to choose some action $a \neq a^*$ at $h$ in the original game, then, in the new game $\Gamma'$, we ask agent $i$ to choose not only her current action, but all future actions that she would have chosen.
according to $S^m_i(\cdot)$ as well. By doing so, we have created a new game in which every action (except for $a^*$, if it exists) at $h$ clinches some payoff $x$, and further, agent $i$ is never called upon to move again.\footnote{More precisely, all of $i$’s future moves are trivial moves in which she has only one possible action; hence these histories may further be removed to create an equivalent game in which $i$ is never called on to move again. Note that this only applies to the actions $a \neq a^*$; it is still possible for $i$ to follow $a^*$ at $h$ and be called upon to make a non-trivial move again later in the game.}

We construct strategies in $\Gamma'$ that are the counterparts of strategies from $\Gamma$, so that for all agents $j \neq i$, they continue to follow the same action at every history as they did in the original game, and for $i$, at history $h$ in the new game, she takes the action $a_m$ that is associated with the strategy $S^m_i$ in the original game. By definition if all the agents follow strategies in the new game analogous to the their strategies from the original game, the same terminal history will be reached, and so $\Gamma$ and $\Gamma'$ are equivalent under their respective strategy profiles.

We must also show that if a strategy profile is obviously dominant for $\Gamma$, this modified strategy profile is obviously dominant for $\Gamma'$. To see why the modified strategy profile is obviously dominant for $i$, note that if her obviously dominant action in the original game was part of a strategy that guarantees some payoff $x$, she now is able to clinch $x$ immediately, which is clearly obviously dominant; if her obviously dominant strategy was to follow a strategy that did not guarantee some payoff $x$ at $h$, this strategy must have directed $i$ to follow $a^*$ at $h$. However, in $\Gamma'$, the subgame following $a^*$ is unchanged relative to $\Gamma$, and so $i$ is able to perfectly replicate this strategy, which obviously dominates following any of the clinching actions at $h$ in $\Gamma'$. In addition, the game is also obviously strategy-proof for all $j \neq i$ because, prior to $h$, the set of possible payoffs for $j$ is unchanged, while for any history succeeding $h$ where $j$ is to move, having $i$ make all of her choices earlier in the game only shrinks the set of possible outcomes for $j$, in the set inclusion sense. When the set of possible outcomes shrinks, the best possible payoff from any given strategy only decreases (according to $j$’s preferences) and the worst possible payoff only increases, and so, if a strategy was obviously dominant in the original game, it will continue to be so in the new game. Repeating this process for every history $h$, we are left with a new game where, at each history, there are only clinching actions plus (possibly) one passing action, and further, every payoff that is guaranteeable at $h$ is also clinchable at $h$, and $i$ never moves again following a clinching action. This shows parts (i) and (ii). Part (iii) follows immediately from part (ii).

\begin{lemma}
Let $\Gamma$ be an obviously strategy-proof game that is pruned with respect to the obviously dominant strategy profile $(S_i(\succ_i))_{i \in \mathcal{N}}$ and that satisfies Lemmas 1 and 3. At every
\end{lemma}
history \( h^i \) at which an agent \( i \) moves and every terminal history \( \tilde{h}, \) for every payoff \( z, \) one of conditions (a)-(d) must hold.

**Proof.** Assume not. First, consider the case of a non-terminal history \( h^i \) where \( i \) moves and a payoff \( z \) be such that (a), (b) and (c) do not hold at \( h^i, \) i.e., the following are true:

(a') \( z \notin P_i(h^i) \)

(b') \( z \in P_i(h) \) for all \( h \in \{h': h' \not\subset h^i \text{ and } i_{h'} = i\} \)

(c') \( z \notin C_i^\text{ec}(h^i) \) and

Points (b') and (c') imply that \( z \) is possible at every \( h \not\subset h^i \) where \( i \) is to move, but it is not clinchable at any of them. This implies that for any type of agent \( i \) that ranks \( z \) first, any obviously dominant strategy must have the agent passing at all \( h \not\subset h^i \) where she she is called to move.\(^{58}\)

Towards a contradiction, assume that (d) did not hold, i.e., there exists some \( h' \not\subset h^i \) such that \( i_{h'} = i \) and some \( x \in C_i(h') \) such that \( x \notin C_i(h^i). \) Consider a type \( z \succ_i x \succ_i \cdots. \)

We will show that there is some \( \hat{h}^i \not\subset h^i \) such that type \( \succ_i \) has no obviously dominant action. By the previous paragraph, at any such \( \hat{h}^i \not\subset h^i, \) any obviously dominant strategy must have this type passing. Next, we claim that there must be some \( \hat{h}^i \not\subset h^i \) such that passing is actually not obviously dominant. Since \( z \notin P_i(h^i) \) and \( x \notin C_i(h^i), \) by Lemma 3, the worst case outcome from following this strategy is some \( y \) that is strictly worse than \( x \) according to \( \succ_i. \) However, we also have \( x \in C_i(\hat{h}^i) \) for some \( \hat{h}^i \not\subset h^i, \) and so, the best case outcome from clinching \( x \) at \( \hat{h}^i \) is \( x. \) This implies that passing is not obviously dominant, which contradicts that \( \Gamma \) is OSP.

Last, consider a terminal history \( \tilde{h}. \) As above, let \( z \) be a payoff such that (a'), (b'), and (c') hold. Recall that for terminal histories, we define \( C_i(\tilde{h}) = \{y\} \) for all \( i, \) where \( y \) is the unique outcome that obtains at \( \tilde{h}. \) Assume that (d) does not hold, i.e., there exists some \( h' \not\subset \tilde{h} \) such that \( i_{h'} = i \) and \( x \in C_i(h') \) such that \( x \notin C_i(\tilde{h}). \) Note that (i) \( z \neq y \) (because \( z \notin P_i(\tilde{h}), \) by (a')); (ii) \( z \neq x \) (by (c')); and (iii) \( x \neq y \) (because \( x \notin C_i(\tilde{h}). \)) In other words, \( x, y, z \) must all be distinct payoffs for \( i. \) Consider the type \( z \succ_i x \succ_i y \succ_i \cdots. \) By (b') and (c'), \( z \) is possible at every \( h \not\subset \tilde{h} \) where \( i \) is to move, but is not clinchable at any such history. Thus, any obviously dominant strategy of type \( \succ_i \) must have agent \( i \) passing at any such history. However, at \( h', i \) could have clinched \( x, \) and so passing is not obviously dominant (because \( y \) is possible from passing). \( \square \)

\(^{58}\)At all such \( h, \) since \( z \) is not clinchable, but is possible, it must be possible following the (unique) passing action. This means that best case outcome from passing is \( z, \) while the worst case outcome from clinching is strictly worse than \( z. \) Thus, no clinching action can be obviously dominant, so, if an obviously dominant strategy exists, it must instruct \( i \) to pass.
Proposition 2 follows from Lemmas 1, 3, and 4. Theorem 1 then follows from Propositions 1 and 2.

B.2 Proof of Theorem 4

The proof follows the same steps as the proof of the analogous result for OSP in Li (2017b). We present it for the case of SOSP. Consider any two strategies $S_i$ and $S'_i$ of agent $i$; let $h$ be the earliest point of departure for these two strategies.

Suppose $S_i$ is strongly obviously strategy-proof in $\Gamma$. Then any outcome that is possible after playing $S_i$ is weakly better than any outcome that is possible after playing $S'_i$ in game $\Gamma'$, and hence in the outcome set-equivalent game $\Gamma'$. Hence, $S_i$ strongly obviously dominates $S'_i$ in game $\Gamma'$, and thus it weakly dominates it.

Now suppose $S_i$ weakly dominates $S'_i$ in all games $\Gamma'$ that are outcome equivalent to $\Gamma$. Consider such a $\Gamma'$ in which all moves of agent $i$ following history $h$ are made by Nature instead. Since, $S_i$ weakly dominates $S'_i$ in $\Gamma'$, we conclude that any outcome that is possible after playing $S_i$ is weakly better than any outcome that is possible after playing $S'_i$ in game $\Gamma'$, and hence in the outcome set-equivalent game $\Gamma$. Hence, $S_i$ strongly obviously dominates $S'_i$ in game $\Gamma$. ■

B.3 Proof of Theorem 6

($\Leftarrow$) Let $\Gamma$ be a monotonic millipede game. Fix an agent $i$, and, for any history $h^*$ at which $i$ moves, let $\bar{x}_{h^*} = \text{Top}(\succ_i, P_i(h^*))$ and $\tilde{y}_{h^*} = \text{Top}(\succ_i, C_i(h^*))$. Let $H_{i,h^*} = \{ h \in H_i(h^*) | h^* \subseteq h' \subseteq h \implies h' \notin H_i \}$ be the set of one-step-foresight simple nodes. Consider the following strategic plan for any $h^*$:

- If $\bar{x}_{h^*} \in C_i(h^*)$, then $S_{i,h^*}(h^*) = a_{\bar{x}_{h^*}}$, where $a_{\bar{x}_{h^*}} \in A(h^*)$ is a clinching action for $\bar{x}_{h^*}$.

- If $\bar{x}_{h^*} \notin C_i(h^*)$, then $S_{i,h^*}(h^*) = a^* (i$ passes at $h^*)$, and, for any other $h \in H_{i,h^*}$:
  
  - If $P_i(h^* \setminus C_i(h^*)) \subseteq C_i(h)$, then $S_{i,h^*}(h^*) = a_{\bar{x}_{h^*}}$.
  - If $C_i(h^* \setminus C_i(h)) \subseteq C_i(h)$, then $S_{i,h^*}(h^*) = a_{\tilde{y}_{h^*}}$.

Note that by monotonicity, at any $h \in H_{i,h^*}$, one of the conditions in the last two bullet points must hold. It is straightforward to verify that this strategic plan is OSF-simple at any $h^*$, and thus the corresponding strategic collection $(S_{i,h^*})_{h^* \in H_i}$ is also OSF-simple.

($\Rightarrow$) Let $\Gamma$ be a millipede game that is not monotonic, which means there exists an agent $i$, a history $h^*$ at which $i$ moves, another history $h \in H_{i,h^*}$, and payoffs $x$ and $y$ such that $x \in (P_i(h^* \setminus C_i(h^*)) \setminus C_i(h)$ and $y \in C_i(h^*) \setminus C_i(h)$. Notice that $x \neq y$. Without
loss of generality we assume that \( h \) is the earliest history at which monotonicity is violated in this way.\(^{59}\) Since both \( x, y \notin C_i(h) \) by definition, there is some third payoff \( z \neq x, y \) such that \( z \in C_i(h) \). Let \( \succ_i \) be a type of agent \( i \) such that \( \text{Top}(\succ_i, P_i(h^*)) = x \) and \( \text{Top}(\succ_i, P_i(h^*) \setminus \{x\}) = y \), and let \( \succ_i' \) be a type of agent such that \( \text{Top}(\succ_i', P_i(h^*)) = x \) and \( \text{Top}(\succ_i', P_i(h^*) \setminus \{x\}) = z \). Note that for both \( \succ_i \) and \( \succ_i' \), for any OSF-dominant plan \( S_{i,h^*}(h^*) = a^* \) (because \( x \) is possible, but not clinchable at \( h^* \)).

There are two cases, depending on what is possible at \( h \).

**Case (1):** \( y \notin P_i(h) \). From above, \( S_{i,h^*}(h^*) = a^* \). However, for any such strategic plan, the worst case outcome from the perspective of \( h^* \) is some \( w \neq x, y \).\(^{60}\) Since she can clinch \( y \) at \( h^* \), and \( y \succ_i w \), \( S_{i,h^*}(\cdot) \) is not OSF-dominant.

**Case (2):** \( y \in P_i(h) \). Here, there are two subcases.

**Subcase (2).(i):** \( z \in P_i((h, a^*)) \). In this case, type \( \succ_i \) has no OSF-dominant strategic plan at \( h^* \). Again, in any such plan, we have \( S_{i,h^*}(h) = a^* \). But, since \( z \in P_i((h, a^*)) \), for any \( a \in A(h) \), the worst case from the perspective of node \( h^* \) is at best \( z \) (since both \( x, y \notin C_i(h) \), by definition), which is worse than clinching \( y \) at \( h^* \), and so \( S_{i,h^*}(\cdot) \) is not OSF-dominant.

**Subcase (2).(ii):** \( z \notin P_i((h, a^*)) \). If \( x \in P_i(h) \), then type \( \succ_i' \) has no OSF-dominant strategic plan at \( h \).\(^{61}\) To see this, note that at \( h \), for any strategic plan, the worst case from passing at \( h \) is strictly worse than \( z \) (since \( x \) is possible, but not clinchable at \( h \), and \( z \notin P_i((h, a^*)) \), while \( z \in C_i(h) \), and so \( S_{i,h}(h) \neq a^* \). However, \( S_{i,h}(h) \) must equal \( a^* \) because the best case from passing is \( x \); a contradiction.

If \( x \notin P_i(h) \), then type \( \succ_i \) has no OSF-dominant strategic plan at \( h^* \).\(^{62}\) Once again, any such plan must have \( S_{i,h^*}(h^*) = a^* \). Since \( y \in P_i(h) \) but \( y \notin C_i(h) \), it must be that \( y \in P_i((h, a^*)) \), and so there is some other \( w \neq x, y \) such that \( w \in P_i((h, a^*)) \) (because \( x \notin P_i(h) \)). Therefore, from the perspective of node \( h^* \), for any fixed plan \( S_{i,h^*}(h) \), the worst case is at best \( w \), which is strictly worse than clinching \( y \) at \( h^* \), and thus \( S_{i,h^*}(\cdot) \) is not OSF-dominant.

\(^{59}\)This assumption guarantees that, in a pruned game, history \( h^* \) is on path of the play for the two agent types we construct.

\(^{60}\)If \( x \notin P_i(h) \), then this is obvious (since in this case, \( y \notin P_i(h) \) either). If \( x \in P_i(h) \), by definition \( x \notin C_i(h) \), and so \( x \) is only possible following a pass at \( h \), \( x \in P_i((h, a^*)) \). By definition of a passing action, there is some other \( w \neq x \) such that \( w \in P_i((h, a^*)) \). Since \( y \notin P_i(h) \), \( w \neq y \).

\(^{61}\)Note that here, we consider \( h \), not \( h^* \); in fact, the argument actually shows something stronger, which is that there is no obviously dominant action at \( h \), and so the game in this case is not OSP, let alone OSF-dominant.

\(^{62}\)As with the previous case, the statement can actually be made stronger: there is actually no obviously dominant action at \( h^* \).
B.4 Proof of Theorem 7

Because of the restriction to paths of the game that are on the path of the greedy strategies for some type profile, it is sufficient to prove this theorem for pruned games. We first note the following lemma, which says that the first time an agent is called to play in a pruned SOSP game, all of her actions are associated with a unique undominated payoff, except for possibly one action, which may have two undominated payoffs. To state the lemma, define \( \tilde{P}_i(h) = \{ x \in P_i(h) : \nexists y \in P_i(h) \text{ s.t. } y \succ x \} \) to be the set of possible payoffs for \( i \) at \( h \) that are undominated.

**Lemma 5.** Let \( \Gamma \) be a pruned SOSP game. Let \( h_0^i \) be any earliest history at which agent \( i \) is called to play. Then, \( |\tilde{P}_i((h_0^i, a))| \leq 2 \) for all \( a \in A(h_0^i) \), with equality for at most one \( a \in A(h_0^i) \).

**Proof of lemma.** Since \( h_0^i \) is the first time \( i \) is called to move, it is on-path for all types of agent \( i \). We first show that \( |\tilde{P}_i((h_0^i, a))| \leq 2 \) for all \( a \in A(h_0^i) \). Assume not, which means that there exists some \( a \in A(h_0^i) \) such that \( |\tilde{P}_i((h_0^i, a))| \geq 3 \). Let \( x, y, z \in \tilde{P}_i((h_0^i, a)) \) be three distinct undominated payoffs that are possible following \( a \). By pruning, there must be some type, \( \succ_i \), such that action \( a \) is strongly obviously dominant at \( h_0^i \). Without loss of generality, let \( Top(\succ_i, P_i(h_0^i)) = x \).

Now, note that the worst case from action \( a \) is strictly worse than \( x \) (since \( y, z \) are possible). Again, without loss of generality, assume that \( x \succ_i y \succ_i z \) (such a type exists by richness and the assumption that \( x, y, z \) are all undominated at \( h_0^i \)). For \( a \) to be strongly obviously dominant, for all other \( a' \neq a \), the best case outcome for type \( \succ_i \) must be no better than \( z \); in particular, this implies that for all \( a' \neq a \) and all \( w \in P_i((h_0^i, a')) \), \( w \nleq y \) (note that since \( \geq \) is reflexive, this includes \( y \) itself). Choose some \( w \in P_i((h_0^i, a')) \) for some \( a' \neq a \), and consider a type such that \( Top(\succ_i', P_i(h_0^i)) = y \) and \( y \succ_i' w \succ_i' x \). For this type, the worst case from \( a \) is at best \( x \), while the best case from \( a' \) is \( w \), so \( a \) is not strongly obviously dominant; for any \( a' \neq a \), the worst case is strictly worse than \( y \) (since nothing that dominates \( y \) is possible following any \( a' \neq a \)), while the best case from \( a \) is \( y \), and so no \( a' \neq a \) is strongly obviously dominant either. Therefore, type \( \succ_i' \) has no strongly obviously dominant action, which is a contradiction.

Finally, we show that \( |\tilde{P}_i((h_0^i, a))| = 2 \) for at most one \( a \in A(h_0^i) \). Let \( a \) and \( a' \) be two actions such that there are two possible undominated payoffs for \( i \) following each, and, for notational purposes, let \( \tilde{P}_i((h_0^i, a)) = \{ x, y \} \). Again, by pruning, there is some type \( \succ_i \),

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63 By definition, \( Top(\succ_i, P_i(h_0^i)) \) must be some undominated payoff that is possible at \( h_0^i \), and it is without loss of generality to assume it is \( x \).

64 If \( x \succeq w \) for all \( w \in P_i((h_0^i, a')) \) for all \( a' \neq a \), then we consider a type such that \( y \succ_i' w \succ_i' z \), and make the same argument. Note that these types exist by richness and the fact that \( x, y, z \) are all mutually undominated.

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that selects action $a$ as a strongly obviously dominant action, and, as above, without loss of generality, let $\text{Top}(\succ_i, P_i(h_0)) = x$. Since $y$ is possible following $a$, in order for $a$ to be strongly obviously dominant, the best case from any $a' \neq a$ must be no better than $y$; in other words, for all $w \in P_i((h_0, a'))$, $w \not\succ x$ (including $w = x$ itself). Therefore, let $\hat{P}_i((h_0, a')) = \{w, z\}$, where, as just argued, $w, z \not\succ x$. It is also without loss of generality to assume that $y$ and $z$ do not dominate each other (since by supposition there are two undominated payoffs following $a'$, and at most one of them can be related to $y$ via dominance). If $w \succeq y$, then consider a specific type such that $\text{Top}(\succ_i, P_i(h_0)) = x$ and $x \succ_i z \succ_i w$ (which one again exists by richness). Since nothing that dominates $x$ is possible following any $a' \neq a$ (including $x$ itself), no such $a'$ can be strongly obviously dominant for this type. Further, the worst case from $a$ is at best $y$, while the best case from $a'$ is $z \succ_i y$, and so $a$ is also not strongly obviously dominant. Therefore, this type has no strongly obviously dominant action. If $w \not\succeq y$, then consider a type such that $x \succ_i z \succ_i y \succ_i w$, and once again note that this type has no strongly obviously dominant action at $h$. ■

Continuing with the main proof, if a history $h$ is payoff-relevant, then by definition $|P_i(h)| \geq 2$. Assume that there was a path of the game with two payoff-relevant histories $h_1 \subset h_2$ for some agent $i$, and note that it is without loss of generality to assume that $h_1$ and $h_2$ are the first and second times $i$ is called to play on the path, and that $|P_i((h_1, a))| > 1$ for some $a \in A(h_1)$. In light of the previous lemma that $|\hat{P}_i((h_1, a))| \leq 2$ for all $a \in A(h_1)$, with equality for at most one $a$, there are two cases.

Case (1): There exists payoff relevant histories $h_1 \subset h_2$ such that $|\hat{P}_i((h_1, a))| = 2$, where $a$ is the unique action such that $(h_1, a) \subset h_2$.

In this case, action $a$ has two undominated possible payoffs. By the previous lemma, there can only be one such action, which we will denote $a_i^*$. For notational purposes, define $\hat{P}_i((h_1, a_i^*)) = \{x, y\}$, where $x$ and $y$ are both undominated payoffs. By pruning, there must be some type whose obviously dominant strategy selects $a_i^*$; without loss of generality, let $\text{Top}(\succ_i, P_i(h_1, a_i^*)) = x$.

Next, we claim that for all $a' \neq a_i^*$ and all $w \in P_i((h_1, a'))$, we have $w \succeq y$. To see this, assume that there was some such $a'$ and $w$ such that $w \not\succeq y$. By the previous lemma, $w \not\succeq x$ for all $w \in P_i(h_1)$.\textsuperscript{65} If $y \succeq w$, then $y \succ w$ (since $w \not\succeq y$). By pruning, some type $\succ_i$ is selecting action $a'$, and it is strongly obviously dominant; however, the worst case from $a'$ is at best $w$, while $y$ is possible from $a$. Since $y \succ w$, we have $y \succ_i w$, and so $a'$ does not strongly obviously dominate $a$, which is a contradiction. If $y \not\succeq w$, then neither $y$ nor $w$ dominate each other.

\textsuperscript{65}If $w \succeq x$ for some $w \in P_i((h_1, a'))$, then, by the lemma, $\hat{P}_i((h, a')) = \{\hat{w}\}$ and $\hat{w}$ dominates all other $w$ that are possible following $a'$ (including $x$). But, this contradicts that $a_i^*$ was strongly obviously dominant for type $\succ_i$. 

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and type \( x \succ_i w \succ_i y \) has no strongly obviously dominant action at \( h_1 \). Therefore, \( w \succeq y \) for all \( w \in P_i((h_1, a')) \) and all \( a' \neq a_i \). In fact, it is further the case that \( w = y \); to see this, note that if there exists some \( w \succ y \), then type \( x \succ_i w \succ_i y \) again has no strongly obviously dominant action.\(^{66}\) Thus, we have shown that for all \( a' \neq a_i^*, P_i((h_1, a')) = \{y\} \).

Since \( h_2 \) is payoff relevant, there must exist some \( x', y' \in \hat{P}_i(h_2) \) that are undominated.\(^{67}\) Further, we claim that \( y' = y \) and \( x' = x \). To see the former, first note that \( y \in \hat{P}_i((h_1, a_i^*)) \) (i.e., \( y \) is undominated at \( (h_1, a_i^*) \)), and so we cannot have \( y' \succ y \). If \( y \npreceq y' \), then type \( x \succ_i y \succ_i y' \) has no strongly obviously dominant action at \( h_1 \). Finally, to see that \( x' = x \), again note that we cannot have \( x' \npreceq x \); if \( x \npreceq x' \), then type \( x \succ_i y \succ_i x' \) has no strongly obviously dominant action at \( h_1 \). Thus, \( \hat{P}_i(h_2) = \{x, y\} \).

Finally, note that any type that prefers \( x \succ_i y \) must select action \( a_i^* \) at \( h_1 \), and thus, \( h_2 \) is on-path; further, any type that prefers \( y \succ_i x \) must select some \( a' \neq a_i^* \) at \( h_1 \); in other words, for all types that reach \( h_2 \), \( \text{Top}(\succ_i, P_i(h_2)) = x \). Recall that \( \left| A(h_2) \right| \geq 2 \), and there must be at least one action such that \( y \) is a possible outcome. Label this latter action \( a_2 \) (i.e., \( y \in P_i((h_2, a_2)) \)), and let \( a'_2 \) be some other action. By pruning, there must exist some types \( \succ_i \) and \( \succ'_i \) whose strongly obviously dominant strategies select \( a_2 \) and \( a'_2 \), respectively. But, \( \text{Top}(\succ_i, P_i(h_2)) = \text{Top}(\succ'_i, P_i(h_2)) = x \) (indeed, as just argued above, \( x \) is the top choice for all types of \( i \) that reach \( h_2 \)), which implies that \( x \in P_i((h_2, a_2)) \) and \( x \in P_i((h_2, a'_2)) \). However, \( y \in P_i((h_2, a_2)) \), and so \( a_2 \) is not strongly obviously dominant for type \( \succ_i \), which is a contradiction.

**Case (2):** For all payoff-relevant histories \( h_1 \subset h_2, |\hat{P}_i((h_1, a))| = 1 \), where \( a \) is the unique action such that \( (h_1, a) \subset h_2 \).

Note that \( |\hat{P}_i((h_1, a))| = 1 \) implies that \( |\hat{P}_i(h_2)| = 1 \). Let \( \hat{P}_i(h_2) = \{x\} \), and note that by definition, \( x \succeq x' \) for all \( x' \in P_i(h_2) \), which implies that for all types of agent \( i \), \( x \succ_i x' \) for all \( x' \in P_i(h_2) \). Since there are no trivial moves, \( \left| A(h_2) \right| \geq 2 \). Since \( h_2 \) is payoff-relevant, \( |P_i(h_2)| \geq 2 \), i.e., there must exist some \( a'_2 \in A(h_2) \) and \( x' \in P_i((h_2, a'_2)) \) such that \( x' \neq x \). Further, by pruning, there is some type that has a strongly obviously dominant strategy that selects \( a'_2 \). This implies that, for any \( a_2 \neq a'_2, x \notin P_i((h_2, a_2)) \), and so \( x \in P_i((h_2, a'_2)) \). Again by pruning, there must be some type \( \succ_i \) that has a a strongly obviously dominant strategy that selects \( a_2 \). But, as just argued, \( x \notin P_i((h_2, a_2)) \) and \( x \in P_i((h_2, a'_2)) \). Since all types are such that \( \text{Top}(\succ_i, P_i(h_2)) = x \), \( a_2 \) is not strongly obviously dominant, which is a contradiction.

\(^{66}\)The worst case from \( a_i^* \) is at best \( y \), while \( w \) is possible from some \( a' \), and so \( a_i^* \) is not strongly obviously dominant; similarly, the worst case from any \( a' \neq a_i^* \) is strictly worse than \( x \), while the best case from \( a_i^* \) is \( x \).

\(^{67}\)If not, then all payoffs in \( P_i(h_2) \) can be ordered by the dominance relation \( \succeq \), and, if a strongly obviously dominant action exists, all types will take the same action, and the remaining actions can be pruned.
B.5 Proof of Theorem 8

That sequential price mechanisms are SOSP is immediate from the definition, and so we focus on proving that every SOSP game is equivalent to a sequential price mechanism. Note first that the pruning principle continues to apply to strong obvious dominance. Also, following the same reasoning as in the proof of Theorem 1, given any SOSP game, we can construct an equivalent SOSP game of perfect information in which Nature moves at most once, as the first mover, and so we can focus on the deterministic subgame after any potential move by Nature. Thus, what remains to show is that every perfect-information, pruned SOSP game in which there are no moves by Nature is equivalent to a sequential price mechanism.

Let \( \Gamma \) be such a game. By Theorem 7, each agent \( i \) can have at most one payoff-relevant history along any path of game \( \Gamma \), and this history (if it exists) is the first time \( i \) is called to play. Consider any such history \( h_i^0 \). If there is some other history \( h' \supset h_i^0 \) at which \( i \) is called to play, then history \( h' \) must be payoff-irrelevant for \( i \); in other words, there is some payoff \( x \) such that \( P_i((h', a')) = \{x\} \) for all \( a' \in A(h') \). Using the same technique as in the proof of Theorem 1, we can construct an equivalent game \( \Gamma' \) in which at history \( h_i^0 \), \( i \) is asked to also choose her actions for all successor histories \( h' \supset h_i^0 \) at which she might be called to play, and then is not called to play again after \( h_i^0 \) (see the proof of Theorem 1 for a more formal description of this procedure). Since all of these future histories were payoff-irrelevant for \( i \), the new game continues to be strongly obvious dominant for \( i \). Strong obvious dominance is also preserved for all \( j \neq i \), since having \( i \) make all of her choices earlier only shrinks the set of possible outcomes any time \( j \) is called to move, and thus, if some action was strongly obviously dominant in the old game, the analogous action(s) will be strongly obviously dominant in the new game. Repeating this for every agent and every history, we have constructed a SOSP game \( \Gamma' \) that is equivalent to \( \Gamma \) and in which each agent is called to move at most once along any path of play.

We claim that \( \Gamma' \) is a sequential price mechanism. Assume not, and let \( h \) be an earliest history where the definition of a sequential price mechanism is violated. Since \( \Gamma' \) is not a sequential price mechanism, there must be some payoff \( x \in P_i(h) \) that \( i \) cannot clinch at \( h \). Note that it is without loss of generality to assume that there exists such an unclinchable \( x \) that is not dominated, i.e., \( x \in \hat{P}_i(h) \). Since \( x \) is not clinchable, for any action \( a \in A(h) \) such that \( x \in P_i((h, a)) \), there is some other \( y \in P_i((h, a)) \).

**Case (1):** \( |P_i(h)| \geq 3 \) and there exists a \( y \in P_i((h, a)) \) such that \( x \) and \( y \) do not dominate each other.

By Lemma 5 (in the proof of Theorem 7), \( a \) is the unique action such that \( |\hat{P}_i((h, a))| = 68 \) If all \( x' \in \hat{P}_i(h) \) are clinchable at \( h \), then all types will be able to take an action that clinches their top possible payoff, and any other action can be pruned.
\{x, y\}, and, for any other \(a' \neq a\), let \(\hat{P}_i((h, a')) = \{w'\}\). We first claim that for any \(a' \neq a\), \(\hat{P}_i((h, a')) = \{y\}\).

Assume not, i.e., there exists some \(a' \neq a\) and \(w' \neq y\) such that \(\hat{P}_i((h, a')) = \{w'\}\). First, since \(x\) is not clinchable, any type such that Top(\(\succ_i, P_i(h)\)) = \(x\) must select \(a\), and \(x \notin P_i((h, a'))\) for any \(a' \neq a\). Now, if \(x \succ w'\), then type \(x \succ_i w' \succ_i y\) has no strongly obviously dominant action at \(h\); therefore, \(x \not\succ w'\). If \(y \succ w'\), then \(y \notin P_i((h, a'))\) (since by assumption \(w'\) is undominated at \((h, a')\)); however, if this is the case, then it is not strongly obviously dominant for any type to select \(a'\) (since \(y \in P_i((h, a'))\)), and it can be pruned. Therefore, \(y \not\succ w'\). If \(w' \succ y\), then, once again, type \(x \succ_i w' \succ_i y\) has no strongly obviously dominant action at \(h\). Therefore, \(w' \not\succ y\). Thus, the only remaining possibility is that \(x, y, w' \in \hat{P}_i(h)\), i.e., \(x, y, w'\) are all mutually undominated payoffs at \(h\). But then, type \(x \succ_i w' \succ_i y\) has no strongly obviously dominant action at \(h\). Therefore, \(\hat{P}_i((h, a')) = \{y\}\) for all \(a' \neq a\).

We also claim further that \(P_i((h, a')) = \{y\}\) for all \(a' \neq a\); indeed, if this were not the case, then there is some \(a'\) and some \(w' \in P_i((h, a'))\) such that \(y \succ w'\). By pruning, some type \(\succ_i\) must be selecting action \(a'\). However, the worst case from \(a'\) (for all types) is at best \(w'\), while \(y\) is possible following \(a\), and so \(a'\) is not strongly obviously dominant for type \(\succ_i\). Therefore, \(P_i((h, a')) = \{y\}\) for all \(a' \neq a\).

Let \(z \neq x, y\) be some third payoff that is possible at \(h\). In light of the previous paragraph, \(z \in P_i((h, a))\), and \(z \notin P_i((h, a'))\) for all other \(a' \neq a\). Finally, note that type \(x \succ_i y \succ_i z\) has no strongly obviously dominant action at \(h\). (Note that since \(\hat{P}_i(h) = \{x, y\}\), \(z\) must be dominated by one of \(x\) or \(y\), and so by our richness assumption, such a type exists.)

**Case (2):** \(|P_i(h)| \geq 3\) and for all other \(y \in P_i((h, a)), x \succ y\).

Since \(i\) cannot clinch \(x\) at \(h\), we have that, for all other \(a'\) and \(w \in P_i((h, a'))\), \(x \succ y \succeq w\). By pruning, some type \(\succ_i\) must be choosing action \(a'\); however, the previous sentence implies that \(a'\) is not strongly obviously dominant for this type, which is a contradiction.

**Case (3):** \(|P_i(h)| = 2\).\(^{71}\)

Let \(P_i(h) = \{x, y\}\). Given the definition of a sequential price mechanism, the only case we need to rule out is that neither \(x\) nor \(y\) is clinchable, i.e., they are at least two actions in \(A(h)\), and, for all \(a \in A(h), P_i((h, a)) = \{x, y\}\). At least one of \(x \succ_i y\) or \(y \succ_i x\) must hold for some type that reaches \(h\); however, it is simple to see that no matter which type reaches \(h\), they will not have a strongly obviously dominant action.

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\(^{69}\)Recall that \(x\) is not dominated at \(h\), so such a type does indeed exist.

\(^{70}\)Note that in particular, this implies that \(x \succeq w\) for all \(w \in P_i(h)\). Indeed, if there were some \(w \in P_i(h)\) such that \(w\) and \(x\) did not dominate each other, then type \(x \succ_i w \succ_i y\) has no strongly obviously dominant action at \(h\). (Recall also that \(x\) is undominated at \(h\), and so there is no \(w \succ x\), either.)

\(^{71}\)If \(|P_i(h)| = 1\), the argument is trivial.
B.6 Proof of Theorem 9

B.6.1 Roles and “Symmetry to Symmetrization” Reduction Lemma

For clarity of the exposition, it is convenient to sometimes distinguish between an agent \( i \) moving at some set of histories and a “role” moving at these same histories.\(^{72}\) Formally, we create a copy \( \mathcal{R} \) of the set of agents \( \mathcal{N} \). Given a perfect-information mechanism \((\Gamma, S)\) we create a copy of the game \( \Gamma \) as a game between these roles—treated as agents—and we create a copy of the strategy profile \( S \) as strategies of these roles. With some abuse of notation we refer to the copy of \((\Gamma, S)\) by the same symbols. For a game \( \Gamma \), the function \( \rho : \mathcal{H} \rightarrow \mathcal{R} \) maps each history \( h \) to the role \( \rho(h) \) that moves at this history.

We use the role copy of \((\Gamma, S)\) to create mechanisms \((\Gamma_\sigma, S_\sigma)\) that differ only in the mapping of agents (and their preferences) to the roles. The preferences of the roles are determined by the preferences of the original agents and a bijection \( \sigma : \mathcal{R} \rightarrow \mathcal{N} \). We call this bijection a **role assignment** function, and we denote by \( \Sigma \) the space of all role assignment functions. We define \( \Gamma_\sigma \) as the extensive-form game with the same game tree as \( \Gamma \) and such that at each non-terminal history \( h \), the agent called to move is \( \sigma(\rho(h)) \); at each terminal history in \( \Gamma_\sigma \) the payoff of agent \( \sigma(i) \) is the same as the payoff of \( i \) at the corresponding history in \( \Gamma \). The strategy \( S_\sigma \) of agent \( \sigma(i) \) is the same as the strategy of agent \( i \) in the original game \( \Gamma \). There are \( |\Sigma| = N! \) possible mechanisms \((\Gamma_\sigma, S_\sigma)\); we call them the permuted mechanisms.

We further define the **symmetrized mechanism** \((\Gamma^*, S^*)\) to be the following random mechanism: first, Nature chooses a role assignment function \( \sigma \) uniformly at random from the set of all possible role assignment functions, and then, the agents play \( \Gamma_\sigma \) with strategies \( S_\sigma \).\(^{73}\) To formally ensure that the symmetrizations of a millipede is a millipede, we assume that Nature draws the role assignment \( \sigma \) and the path in the subgame \( \Gamma_\sigma \) in the same move.

The following lemma shows that it is sufficient to prove Theorem 9 for symmetrized mechanisms.

**Lemma 6.** Suppose that every symmetrization of a deterministic OSP and Pareto-efficient perfect-information mechanism is equivalent to Random Priority. Then, every symmetric, OSP and Pareto-efficient mechanism is equivalent to Random Priority.

\(^{72}\)Our construction of roles in general extensive-form games extends the role concept from Carroll (2014), who studied them in the (static) context of Pápai (2000)’s hierarchical exchange mechanisms.

\(^{73}\)While this construction implies that different agents play the same strategies in the same role, our arguments only rely on the weaker assumption that an agent’s strategy \( S_{\sigma,i}(\succ_i) \) depends only on her own preferences and her role assignment, and not on the roles assigned to other agents. In other words, in any two subgames \( \Gamma_A \) and \( \Gamma_B \) following Nature’s selection of role assignments \( \sigma_A \) and \( \sigma_B \), if \( \sigma_A^{-1}(i) = \sigma_B^{-1}(i) = r_n \), then \( S^*_A(\succ_i)(h_A) = S^*_B(\succ_i)(h_B) \) for any equivalent histories \( h_A \) and \( h_B \) in these two games.
Proof. Take a symmetric, OSP, and Pareto-efficient mechanism \((\Gamma, S)\). By Lemma 1, we can assume that \((\Gamma, S)\) has perfect information and that Nature moves only at the beginning of the game. Because \((\Gamma, S)\) is symmetric the symmetrized mechanism \((\Gamma^*, S^*)\) is equivalent to \((\Gamma, S)\). Furthermore, \((\Gamma^*, S^*)\) is a lottery over symmetrizations of each deterministic perfect-information continuation game \(\Gamma'\) after Nature’s move in \((\Gamma, S)\). The mechanism given by game \(\Gamma'\), together with the strategy profile induced from \(\Gamma\), is OSP and Pareto efficient, and hence by the assumption of the lemma it is equivalent to Random Priority. Because every lottery over Random Priority lotteries is still equivalent to Random Priority, the lemma obtains. ■

In light of the above lemma, in the sequel we focus on symmetrized mechanisms.

B.6.2 Plan of the Reminder of the Proof

The reminder of the proof builds on the bijective argument used by Abdulkadiroğlu and Sönmez (1998) to show the equivalence of Random Priority and the Core from Random Endowments (see also Pathak and Sethuraman, 2011 and Carroll, 2014). Throughout, we fix the profile of preferences \(\succ_N\). Given any \(\Gamma\) that is OSP and Pareto efficient, we construct a bijection \(f : \Sigma \rightarrow Ord\) that associates to each role assignment function \(\sigma \in \Sigma\) a total linear order of the agents \(f_\sigma \in Ord\) with the property that game \(\Gamma_\sigma\) results in the same final allocation (matching) \(\mu\) as a serial dictatorship where the first agent called to play is \(f_\sigma(1)\), the second agent called to play is \(f_\sigma(2)\), etc. We then show that the mapping we constructed is a bijection, which proves Theorem 9.

B.6.3 Efficient Millipedes

Given any OSP game \(\Gamma\), we can use the techniques of Lemma 3 to construct an equivalent millipede game which has the following properties (recall from Section 5 that for this part, we use \(\mathcal{X}\) to refer to the set of objects to be allocated to the agents, and use \(x, y, z, etc.\) to refer to individual objects in this set; in the terminology of the previous sections, each object can be identified with a unique “payoff” of an agent):

1. At each history \(h\), there is at most one passing action in \(A(h)\); this action, if it exists, is denoted \(a^* \in A(h)\).

2. For every \(x \in G_i(h)\), there exists a clinching action \(a_x \in A(h)\) that clinches \(x\) for \(i\).

3. As soon as an agent’s top still-possible object is guaranteeable at a history \(h\), she clinches this object at \(h\) (that is, agents follow greedy strategies).
If \( i_h = i \) and \( P_i(h) = G_i(h) \), then \( i \) clinches her payoff immediately at \( h \), and is not called to move at any \( h' \supset h \).

Agent \( i \) is active at \( h \) if she has been previously called to play at some \( h' \subset h \), and further has not yet clinched an object at \( h \). Let \( \mathcal{A}(h) \) denote the set of active agents at \( h \).

In constructing the bijection \( f \), we make use of the concept of a lurker introduced by Bade and Gonczarowski (2017, hereafter BG). Informally, a lurker is an agent who has been offered to clinch all objects that are possible for him except for exactly one, which he is said to “lurk”. If an agent lurks some object \( x \), then the mechanism can infer that \( x \) is his favorite (still available) object, and so it is possible to exclude \( x \) from other agents without violating Pareto efficiency. The role of lurkers is to allow more than two agents to be active at any given point of the game; while there can be an arbitrary number of lurkers, at any point, at most two active agents are non-lurkers.

To formally define lurker, let \( C^g_i(h) = \{ x : x \in C_i(h') \text{ for some } h' \subset h \} \) be the set of objects agent \( i \) has been offered to clinch at some subhistory of \( h \). Similarly, let \( C^c_i(h) = \{ x : x \in C_i(h') \text{ for some } h' \subset h \} \) be the set of objects agent \( i \) has been offered to clinch at some strict subhistory of \( h \). We consider a history \( h \) and an active agent \( i \) who has moved at a strict subhistory of \( h \). Let \( h' \subset h \) be the maximal strict subhistory such that \( i_{h'} = i \). Agent \( i \) is said to be a lurker for object \( x \) at \( h \) if (i) \( x \in P_i(h') \), (ii) \( C^c_i(h') = P_i(h') \setminus \{ x \} \) and (iii) \( x \notin C^g_i(h) \) for any other active \( j \neq i \) that was not already a lurker prior to \( h \).

At any \( h \), we partition the set of active agents as \( \mathcal{A}(h) = \mathcal{L}(h) \cup \mathcal{L}(h) \), where \( \mathcal{L}(h) = \{ \ell_1^h, \ldots, \ell_m^h \} \) is the set of lurkers and \( \mathcal{L}(h) \) is the set of active non-lurkers. With some abuse of notation, we let \( \mathcal{X}(h) \) denote the set of still-available (unclinched) objects at \( h \) (rather than outcomes), and partition this set as \( \mathcal{X}(h) = \mathcal{X}^c(h) \cup \mathcal{X}^g(h) \), where \( \mathcal{X}^c(h) = \{ x_1^h, \ldots, x_{\lambda(h)}^h \} \) is the set of lurked objects and \( \mathcal{X}^g(h) = \mathcal{X}(h) \setminus \mathcal{X}^c(h) \) is the set of unlurked objects at \( h \). Each \( \ell_m^h \) has a unique object that she lurks, \( x_m^h \), and the sets are ordered such that if \( m' < m \), then lurker \( \ell_{m'}^h \) is “older” than lurker \( \ell_m^h \), in the sense that \( \ell_{m'}^h \) first became a lurker for \( x_{m'}^h \) at a strict subhistory of the history at which \( \ell_m^h \) became a lurker for \( x_m^h \).
lurkers/lurked objects continues to grow, until eventually, we reach a history $h$ where some agent $i$ clinches some object $x$. BG show that at most two active agents are non-lurkers at any point (see Lemma 13 below). When $i$ clinches at $h$, this initiates a chain of clinching among the active agents that proceeds as follows:

- If $x \in \mathcal{X}(h)$, each lurker $\ell^h_m \in \mathcal{L}(h)$ is immediately assigned to her lurked object, $x^h_m$.
- If $x = x^h_{m_1}$ for some lurked $x^h_{m_1} \in \mathcal{X}(h)$, then all “older” lurkers $\ell^h_{m'}$ for $m' < m_1$ receive their lurked objects $x^h_{m'}$; since lurker $\ell^h_{m_1}$’s lurked object was taken by $i$, she is offered to clinch anything from the remaining set of unclinched objects, $\mathcal{X}(h) \setminus \{x^h_{1}, \ldots, x^h_{m_1}\}$.
- If $\ell^h_{m_1}$ takes an unlurked object, then all remaining lurkers get their lurked objects; if $\ell^h_{m_1}$ chooses a lurked object $x^h_{m_2}$ for some $m_2 > m_1$, then all “older” unmatched lurkers ($\ell^h_{m'}$ for $m_1 < m' < m_2$) get their lurked objects. Lurker $\ell^h_{m_2}$ gets to choose from $\mathcal{X}(h) \setminus \{x^h_{1}, \ldots, x^h_{m_2}\}$, etc.
- This process is repeated until some lurker $\ell^h_m$ chooses an unlurked object, $y$, at which point all remaining unassigned lurkers are assigned to their lurked objects.
- Finally, if $y \in C^e_j(h)$ for the other active non-lurker $j \in \mathcal{L}(h) \setminus \{i\}$, then $j$ is offered to clinch anything from what remains, $\mathcal{X}(h) \setminus \{y\}$.

Notice that at the end of the above chain of lurker assignments that was initiated at $h$, all but at most one active agent in $\mathcal{A}(h)$ has clinched and is thus removed from the game. If all active agents have been assigned, then the continuation game is just a smaller Pareto efficient millipede game on the remaining sets of unmatched agents and objects, which proceeds in the same way. If there is one active agent left, $j$, that has not clinched, then this continuation game begins with agent $j$ carrying over anything that she has been previously offered to clinch, $C^e_j(h)$, to the next round.

**B.6.4 Constructing the bijection**

We now construct the bijection from role assignment functions into serial dictatorship orderings. We start by providing an ordering algorithm that, for a give game $\Gamma$ and fixed preference profile/strategy, follows the path of the game from the root node $h_0$ to the terminal node $\bar{h}$ and outputs orderings a partial ordering of the agents, denoted $\triangleright$. This ordering

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77Such an agent may or may not exist, but if they do, they are unique.

78Note that the ordering here is unrelated to the dominance relation used to define partitions introduced in Section 2, and the use of the symbol $\triangleright$ here should not be confused with the use of the same symbol to represent the strict part of the dominance relation.
is only partial because agents may “tie”. Ties will be of size at most 2, and any way of breaking these ties to create a total linear order will lead to the same serial dictatorship allocation.

Each role assignment function $\sigma_A \in \Sigma$ induces a game $\Gamma_A$ and an associated partial ordering, $\triangleright_A$, via our ordering algorithm. Running the algorithm on all $N!$ role assignment functions gives $N!$ partial orderings. We will then argue that it is possible to “break ties” consistently in such a way that we recover a bijection $f : \Sigma \rightarrow \text{Ord}$.

The intuitive idea behind constructing the partial order $\triangleright$ is as follows. We start by finding the first agent to clinch some object $x$ (after a series of passes) at some history $h$. This induces a chain of assignments of the active agents $A(h)$ as described above. We create $\triangleright$ by ordering agents who receive lurked objects in order of the “age” of the lurked object they receive, i.e., the agent who receives the “oldest” lurked object is ordered first, etc. After this is done, there are at most 2 active agents who have yet to be ordered, one of whom has clinched an unlurked object, say $y$; if $y$ was previously offered to the remaining active agent, then we add both remaining agents to the order without distinguishing between them, i.e., these two agents tie; if $y$ was not previously offered to the other remaining active agent, then we just add the agent who clinched $y$, and the other active agent (if they exist) carries over their “endowment” (the set $C^x_j(h)$) to the next stage. After clearing this first segment of agents, we continue along the game path and find the first unordered agent to clinch an object, and repeat.

**Ordering Algorithm.** Consider any game path from the root node $h_\emptyset$ to a terminal node $\bar{h}$, which is associated with a unique allocation of objects to agents. Each step $k$ of the algorithm below produces a partial ordering $\triangleright^k$ on the set of agents who are processed in step $k$. At the end of the final step $K$, we concatenate the $K$ components to produce $\triangleright$, the final partial ordering on the set of all agents $\mathcal{N}$.

**Step 1** Find the first object to be clinched along the game path, say $x^1$ at history $h^1$ by agent $i^1$.\footnote{That is, $i_{h^1} = i^1$, and $i^1$ selects a clinching action $a_{x^1} \in A(h^1)$ that clinches $x^1$. Also, by Lemma 15, $i^1 \notin \mathcal{L}(h^1)$.} Let $\mathcal{L}(h^1) = \{\ell_1, \ldots, \ell_{\lambda(h^1)}\}$ be the set of lurkers, and $\mathcal{X}^L(h^1) = \{x_1, \ldots, x_{\lambda(h^1)}\}$ be the set of lurked objects (note that these sets may be empty, in which case skip immediately to step 1.2 below).

1. Let $i_{x_1}$ be the agent who ultimately receives $x_1$, $i_{x_2}$ be the agent who ultimately receives $x_2$, up to $i_{x_{\lambda(h^1)}}$ (note that $i_{x_k}$ is not necessarily the agent who lurks $x_k$ at $h^1$, but the agent who ultimately receives $x_k$ at the allocation associated with $\bar{h}$).
2. Let \( j \in \mathcal{L}(h^1) \cup \{i^1\} \) be the unique agent that is not one of the \( i_{x_k} \) from step 1.1. By construction, \( j \) clinches some unlurked object \( y \in \mathcal{X}^L(h^1) \). In addition, there may be one other active agent \( j' \in \mathcal{A}(h^1) \setminus (\mathcal{L}(h^1) \cup \{i^1\}) \).

(a) If such a \( j' \) exists and \( y \in C^L_{j'}(h^1) \), then define \( \succ^1 \) as:

\[
i_{x_1} \succ^1 i_{x_2} \succ^1 \cdots \succ^1 i_{x_{\lambda(h^1)}} \succ^1 \{j, j'\}
\]

(b) Otherwise, define \( \succ^1 \) as:

\[
i_{x_1} \succ^1 i_{x_2} \succ^1 \cdots \succ^1 i_{x_{\lambda(h^1)}} \succ^1 j
\]

In particular, we do not yet order agent \( j' \).

Step \( k \) Find the first object to be clinched along the game path by an agent that has not yet been ordered, say \( x^k \) at history \( h^k \) by agent \( i^k \). Let \( \mathcal{L}(h^k) = \{\ell_1, \ldots, \ell_{\lambda(h^k)}\} \) be the set of lurkers, and \( \mathcal{X}^L(h^k) = \{x_1, \ldots, x_{\lambda(h^k)}\} \) be the set of lurked objects, and carry out a procedure analogous to that from step 1 to produce the step \( k \) order \( \succ^k \).\(^{80}\)

This produces a collection of partial orderings \((\succ^1, \ldots, \succ^K)\), where each \( \succ^k \) is a partial order on the agents processed in step \( k \). We then create the final \( \succ \) in the natural way: for any two agents \( i, j \) who were processed in the same step \( k \), \( i \succ j \) if and only if \( i \succ^k j \). For any two agents \( i, j \) processed in different steps \( k < k' \), respectively, we order \( i \succ j \).

**Remark.** The output of the ordering algorithm is a partial order, \( \succ \), on \( \mathcal{N} \), the set of agents. If there are two agents \( j \) and \( j' \) such that \( j \not\succ j' \) and \( j' \not\succ j \), then we say \( j \) and \( j' \) tie under \( \succ \). Note that by construction, all ties are of size at most 2, and agents can only tie if they are processed in the same step of the algorithm.

**B.6.5 Completing the proof**

We complete the proof Theorem 9 using three key lemmas relating to properties of the partial orders produced by the ordering algorithm applied to games with different role assignments. The proofs of these lemmas are somewhat involved, and so to streamline the presentation of the main argument, we relegate them to the following subsections.

\(^{80}\)At the end of step \( k - 1 \), there is at most one active agent \( j' \in \mathcal{A}(h^{k-1}) \) that was not ordered in step \( k - 1 \). This agent \( j' \), if she exists, is the active non-lurker other than the non-lurker \( i^{k-1} \) that clinched at \( h^{k-1} \) to initiate the step \( k - 1 \) assignments. Thus, after the step \( k - 1 \) assignments are all made, we are left with a subgame where agent \( j \) carries over her previous endowment, \( C^L_j(h^{k-1}) \). This subgame is again a Pareto efficient millipede game, and so the same structure as the original game, but among only the unmatched agents and unclinched objects after step \( k - 1 \). At the “root node” of this subgame, \( h^0_j \), agent \( j \) is offered to clinch \( C_j(h^0_j) \supseteq C^L_j(h^{k-1}) \). All of the structure and arguments from the previous steps are then repeated.
Take a role assignment function $\sigma$, corresponding game $\Gamma_\sigma$, and the partial ordering $\triangleright_\sigma$ that results from applying the ordering algorithm to $\Gamma_\sigma$. Let $f$ be a total ordering of the agents, where $f(1) = i$ is the first agent, $f(2) = j$ is the second agent, etc. We say that $f$ is consistent with $\triangleright_\sigma$ if, for all $j, j'$: $j \triangleright_\sigma j'$ implies $f(j) < f(j')$. In other words, given some partial ordering $\triangleright_\sigma$, total order $f$ is consistent if there is some possible way to break the ties in $\triangleright_\sigma$ that delivers $f$.

Lemma 7. For any total order $f$ consistent with $\triangleright_\sigma$, a serial dictatorship under agent ordering $f$ results in the same final allocation as $\Gamma_\sigma$.

Let $h_A^k$ be the history that initiates step $k$ of the ordering algorithm when it is applied to game $\Gamma_A$. For instance, $h_A^1 = (h_\emptyset, a^*, \ldots, a^*)$ is a history following a sequence of passes such that agent $i_{h_A^1}$ moves at $h_A^1$ and is the first agent to clinch in the game. This induces a chain of assignments of the agents in $L(h_A^1) \cup \{i_{h_A^1}\}$, plus possibly one other active non-lurker at $h_A^1$, as described above. History $h_A^2 \supset h_A^1$ is then the next time along the game path that an agent who was not ordered in step 1 of the ordering algorithm clinches an object, etc.

Lemma 8. Let $\sigma_A, \sigma_B$ be two role assignment functions, $\Gamma_A$ and $\Gamma_B$ their associated games, and $(\triangleright_A^1, \ldots, \triangleright_A^K)$ and $(\triangleright_B^1, \ldots, \triangleright_B^K)$ the respective partial orderings produced by each step of the ordering algorithm. For all $k$, if $\triangleright_A^k = \triangleright_B^k$, then $h_A^k = h_B^k$, and further, $\sigma_A^{-1}(i) = \sigma_B^{-1}(i)$ for all agents $i$ that are ordered in step $k$ of algorithm.

In particular, Lemma 8 implies the following corollary.

Corollary 3. If $\triangleright_A = \triangleright_B$, then $\sigma_A = \sigma_B$.

Lemma 9. There are no three role assignment functions $\sigma_A, \sigma_B$ and $\sigma_C$ such that the resulting partial orders $\triangleright_A$, $\triangleright_B$ and $\triangleright_C$ take the form\footnote{What is meant here is that $\sigma_A, \sigma_B$, and $\sigma_C$ restricted to the agents $\{i_1, \ldots, i_n, i, j\}$ are all distinct role assignment functions that produce partial orderings $\triangleright_A, \triangleright_B$, and $\triangleright_C$ that begin by ordering agents $i_1, \ldots, i_n$ in the exact same way (possibly with ties), and continue by ordering agents $i, j$ in the manner specified.}:

\[
\begin{align*}
&i_1 \triangleright_A \cdots \triangleright_A i_n \triangleright_A \{i, j\} \cdots \\
&i_1 \triangleright_B \cdots \triangleright_B i_n \triangleright_B i \triangleright_B j \cdots \\
&i_1 \triangleright_C \cdots \triangleright_C i_n \triangleright_C j \triangleright_C i \cdots.
\end{align*}
\]

By Corollary 3 of Lemma 8, the mapping from role assignments $\sigma$ to partial orders $\triangleright_\sigma$ generated by the ordering algorithm is an injection. Lemma 9 shows that we can break all the ties—recursively, coding step by coding step—creating from each $\triangleright_\sigma$ a consistent total order $f_\sigma$ in a way that preserves the injectivity. In this way we obtain an injection from role.
assignments \( \sigma \) to serial dictatorships with orders \( f_\sigma \). Because in this injection the domain of role assignments \( \sigma \) and the range of serial dictatorship orderings \( f_\sigma \) are finite and have equal size, this injection is actually a bijection. It remains to check that the millipede \( \Gamma_\sigma \) generates the same allocation as the serial dictatorship with ordering \( f_\sigma \). This is implied by Lemma 7 because, by definition, each complete order \( f_\sigma \) generated by the tie-breaking in partial order \( \triangleright_\sigma \) is consistent with \( \triangleright_\sigma \). ■

B.6.6 Preliminary Results for the Proofs of the Key Lemmas

Before proving the core Lemmas 7, 8, and 9, we state several preliminary lemmas we will use. Lemmas 10-14 are due to BG (though the versions presented here are simplified to apply to millipede games that satisfy the properties of Lemma 3). Lemmas 15 and 16 are new.

Throughout, we fix a Pareto efficient millipede game \( \Gamma \) that satisfies properties (i)-(iii) of Lemma 3.

**Lemma 10.** (BG Lemma E.11) If an agent has not yet clinched an object at a history \( h \), then \( X^L(h) \subseteq P_i(h) \cup C_i^C(h) \). If \( i \in L(h) \), then \( X^L(h) \subseteq C_i^C(h) \).

**Lemma 11.** (BG Lemma E.14) If \( i \in L(h) \) and \( x_\ell \in C_i^C(h) \) for some \( x_\ell \in X^L(h) \), then \( i = i \), \( P_i(h) = G_i(h) = C_i(h) \), and there is no passing action \( a^* \) in \( A(h) \).

**Lemma 12.** (BG Lemma E.16) Let \( L(h) = \{ \ell_1, \ldots, \ell_{\lambda(h)} \} \) be the set of lurkers at \( h \) and \( X^L(h) = \{ x^h_1, \ldots, x^h_{\lambda(h)} \} \), with \( \ell_1 \) lurking \( x^h_1 \), \( \ell_2 \) lurking \( x^h_2 \), etc., where \( m < m' \) if and only if \( \ell_m \) became a lurker at a strict subhistory of the history at which \( \ell_{m'} \) became a lurker. Then,

1. \( x^h_1, \ldots, x^h_{\lambda(h)} \) are all distinct objects.
2. For all \( m = 1, \ldots, \lambda(h) \), \( P_{\ell_m}(h) = X(h) \setminus \{ x^h_1, \ldots, x^h_{m-1} \} \).

**Lemma 13.** (BG Lemma E.19) For all \( h \), \( |L(h)| \leq 2 \).

**Lemma 14.** (BG Lemma E.18, E.20) Let \( h \) be a history with lurked objects and let \( i_{h'} = t \) be the agent who moves at the maximal superhistory of the form \( h' = (h, a^*, \ldots, a^*) \). Then:

(i) Agent \( t \) is not a lurker at \( h \).

(ii) If \( i_h \neq t \), then \( C_i(h) \cap C_t^C(h) = \emptyset \).

(iii) If \( x_\ell \in P_j(h) \) for some non-lurker \( j \) and lurked object \( x_\ell \in X^L(h) \), then \( j = t \) and \( C_j^C(h') = X(h) \).

The agent \( t \) who moves at \( h' \) is called the **terminator**.

We also prove the following additional lemmas.
Lemma 15. Let $h$ be a history such that $L(h) \neq \emptyset$. For any superhistory $h'$ of the form $h' = (h, a^*, a^*, \ldots, a^*)$, we have $i_{h'} \notin L(h)$.

Lemma 15 has the following key implication: let $h$ be a history with lurkers $L(h)$, and $h' = (h, a^*, \ldots, a^*, a_x)$ be a superhistory such that $x$ is the next object to be clinched (with possibly agents passing in the mean time). Then, the agent that clinches $x$ is not a lurker.

Proof. Let $L(h) = \{\ell^h_1, \ldots, \ell^h_{|L(h)|}\}$ be the set of lurkers at $h$ and $X^L(h) = \{x^h_1, \ldots, x^h_{|L(h)|}\}$ the set of lurked objects. Assume that the statement was false, and let $h'$ be the smallest superhistory of $h$ such that $i_{h'} = \ell^h_m$ for a lurker $\ell^h_m$ (that is, $i_{h''} \notin L(h)$ for all $h' \subseteq h'' \subseteq h'$). Note first that, for any $h''$ such that $h \subseteq h'' \subseteq h'$, we have $i_{h''} = j \in L(h)$, and if there exists some lurked $x^h_m \in C^L_j(h'')$, by BG Lemma 11, there is no passing action at $h''$, which is a contradiction. Therefore, any clinching action $a_y \in A(h'')$ clinches some $y \in X(h) \setminus X^L(h)$, and for all terminal histories $h \supset (h'', a_y)$, each lurker $\ell^h_m \in L(h)$ receives his lurked object $x^h_m$. Finally, consider history $h'$. By BG Lemma 12, for each $\ell^h_m \in L(h)$, $P_{\ell^h_m}(h') = P_{\ell^h_m}(h) = X(h) \setminus \{x^h_1, \ldots, x^h_{m-1}\}$ (note that $h'$ is reached from $h$ via a series of passes, and so $X(h) = X(h')$), and $Top(\succ_{\ell^h_m}, P_{\ell^h_m}(h')) = x^h_m$ for all types $\succ_{\ell^h_m}$ such that $h'$ is on the path of play. Therefore, by pruning and greedy strategies, at $h'$, there is no clinching action $a_x$ for any $x \in P_{\ell^h_m}(h') \setminus \{x^h_m\}$. Thus, the only possibility is that every action $a \in A(h')$ clinches $x^h_m$.

This then implies that $\ell^h_m$ gets $x^h_m$ at all terminal $h \supset h'$. Combining this with the previous statement that $\ell^h_m$ gets $x^h_m$ for all terminal $h \supset (h'', a_y)$ for any $h \subseteq h'' \subseteq h'$ and clinching action $a_y \in A(h'')$, we conclude that $\ell^h_m$ gets $x^h_m$ for all terminal $h \supset h$, which means that $\ell^h_m$ was not a lurker at $h$, which is a contradiction. ■

Lemma 16. Let $h$ be a history such that $L(h) \neq \emptyset$ and $A(h) = L(h) \cup \{i, j\}$, where $i$ and $j$ are active non-lurkers at $h$, and let $y \in X^L(h)$ be an unlurked object at $h$. Further, assume that (i) $i_h = i$ (ii) $y \in C_i(h) \cap C^L_j(h)$, and define $\bar{x} = Top(\succ_j, \bar{X}^L(h))$. Then, $\bar{x} \succ_j y$.

Proof. Let $h' \subseteq h$ be the largest subhistory such that $y \in C_j(h')$, and note that for this history, $P_j(h') = P_j(h)$.\footnote{Note that there cannot be a passing action either: if there were, then, since every history is non-trivial, there must be another action. But, as just argued, there can be no clinching actions for any other $x \neq x^h_m$, and thus there must be a clinching action for $x^h_m$, and the passing action would be pruned.} By construction, $j$ passed at $h'$ when she was offered to clinch $y$. If $P_j(h) \cap X^L(h) = \emptyset$, then, by BG Lemma 14, $j$ is the terminator (i.e., $j = t$), and so by that same lemma, $C_i(h) \cap C^L_j(h) = \emptyset$, which is a contradiction (since $y \in C_i(h) \cap C^L_j(h)$). Therefore, $P_j(h) = X^L(h)$,\footnote{If $P_j(h) \subseteq P_j(h')$ (because some new object became lurked between $h'$ and $h$, and so disappeared as a possibility for $j$), then there must be a more recent subhistory $h'' \supset h'$ where role $j$ was re-offered the opportunity to clinch $y$, by definition of a millipede game (or, more primitively, by OSP).} and so $P_j(h') = X^L(h)$ as well. Since $j$
passes at \( h' \) and \( y \in C_j(h'), \Top(\succ_j, P_j(h')) \succ_j y \). Since \( P_j(h') = \tilde{\mathcal{X}}(h), \Top(\succ_j, P_j(h')) = \Top(\succ_j, \tilde{\mathcal{X}}(h)) = \tilde{x} \succ_j y \), as required. \( \blacksquare \)

### B.6.7 Proofs of the Key Lemmas 7, 8, and 9

In the proofs that follow, we will often make statements referring to generic “roles” in a game form \( \Gamma \), to state properties of \( \Gamma \) that are independent of the specific agent that is assigned to that role. For instance, we previously defined \( C_i(h) \) as the set of outcomes that are clinchable for an agent \( i \) at \( h \). Below, we will sometimes write \( C_r(h) \) to refer to the set of outcomes that are clinchable for the role \( r \in R \) at \( h \), or \( P_r(h) \) for the set of outcomes that are possible for role \( r \). (If the role assignment function is such that \( \sigma(r) = i \), then \( C_i(h) = C_r(h), P_i(h) = P_r(h) \), etc.) Analogously to the sets \( \mathcal{A}(h) \) and \( \mathcal{L}(h) \) for active agents and lurkers at a history \( h \), we write \( \mathcal{A}_R(h) \) for the set of active roles at a history \( h \), and \( \mathcal{L}_R(h) \) for the set of roles that are lurkers at \( h \). When we want to refer to the game form with agents assigned to roles via a specific role assignment function \( \sigma_A \), we will write \( \Gamma_A \). In the proofs, we will often move fluidly between agents and roles; to avoid confusion, we use the notation \( i, j, k \) to refer to specific agents, and the notation \( r, s, t \) to refer to generic roles. Finally, note that while the set of lurkers at any \( h \) may differ depending on the role assignment function, the set of lurked objects (and the order in which they become lurked) depends only on \( h \), and is independent of the specific agent assigned to the role that moves at \( h \).

Unless otherwise specified, when we write the phrase “\( i \) clinches \( x \) at \( h \)” (or similar variants), what is meant is that \( i \) moves at \( h \), takes some clinching action \( a_x \in \mathcal{A}(h) \), and receives object \( x \) at all terminal histories \( \bar{h} \supseteq (h, a_x) \).

Finally, the following remark is simply a restatement of part (iii) of the definition of a lurker, but deserves special emphasis, as it will arise frequently in the arguments below.

**Remark 2.** If an object \( x \) has been offered to an active non-lurker at a history \( h \) (i.e., \( x \in C^L_i(h) \) for some \( i \in \bar{L}(h) \)), then \( x \notin \mathcal{X}(\langle h, a \rangle) \) for any \( a \in \mathcal{A}(h) \).

### Proof of Lemma 7

Let agent \( i^* \) be the first agent to clinch in game \( \Gamma_{\sigma} \), which induces the ordering of the first segment of agents in step 1 of the ordering algorithm. Let \( \mathcal{X}(h^*) = \{x_1, \ldots, x_n\} \) be the set of lurked objects at \( h^* \) (which may be empty).

**Case (1):** \( \mathcal{A}(h) = \mathcal{L}(h) \cup \{i^*\} \).

holding for at most one of the (possibly) two active non-lurkers; see Remark 7.1 of Bade and Gonczarowski (2017).
If \( i^* \) clinches an unlurked object \( y \in \bar{X}^\ell(h^*) \), then, in \( \Gamma_\sigma \), all lurkers get their lurked objects (the oldest lurker \( \ell_1 \) gets \( x_1 \), the second oldest lurker \( \ell_2 \) gets \( x_2 \), etc.), and in the resulting serial dictatorship \( f_\sigma \), the agents are ordered \( f_\sigma : \ell_1, \ell_2, \ldots, \ell_n, i^* \). By BG Lemma 12, for each lurker \( \ell_m \), we have \( x_m = Top(\succ_\ell_m, X \setminus \{x_1, \ldots, x_{m-1}\}) \). When it is agent \( \ell_m \)'s turn in the serial dictatorship, she is offered to choose from \( X \setminus \{x_1, \ldots, x_{m-1}\} \), and thus selects \( x_m \). Finally, consider agent \( i^* \). In game \( \Gamma_\sigma \), when she clinches \( y \) at \( h^* \), it is unlurked. By BG Lemma 10, \( \bar{X}^\ell(h^*) \subseteq P_i(h^*) \cup C_i^\subseteq(h^*) \), which implies that so \( y = Top(\succ_i, \bar{X}^\ell(h^*)) \). At her turn in the serial dictatorship, the set of objects remaining is precisely \( \bar{X}^\ell(h^*) \), and so \( i^* \) will select \( y \).

If, on the other hand, \( i^* \) clinches some lurked object \( x_m \), then all older lurkers \( \ell_1, \ldots, \ell_{m-1} \) get their lurked objects in \( \Gamma_\sigma \), and the resulting serial dictatorship begins as \( f_\sigma : \ell_1, \ldots, \ell_{m-1}, i^* \). By an argument equivalent to the previous paragraph, each of these agents once again gets the same object under the serial dictatorship.\(^{85}\) Then, in \( \Gamma_\sigma \), agent \( \ell_m \) is offered to clinch anything from \( X \setminus \{x_1, \ldots, x_m\} \). If \( \ell_m \) takes another lurked object \( x_{m'} \) for some \( m' > m \), then each lurker \( \ell_{m+1}, \ldots, \ell_{m'-1} \) is assigned to their lurked object, and we add to the serial dictatorship order as \( f_\sigma : \ell_1, \ldots, \ell_{m-1}, i^*, \ell_{m+1}, \ldots, \ell_{m'-1}, \ell_m \). By the same argument as above, at their turn in the resulting SD, each agent \( \ell_{m+1}, \ldots, \ell_{m'-1}, \ell_m \) gets the same object in the SD.\(^{86}\) This process continues until someone eventually takes an unlurked object, all remaining lurkers are ordered, and step 1 is completed.

**Case (2):** \( A(h) = L(h) \cup \{i^*, j\} \) for some \( j \in A(h) \setminus (L(h) \cup \{i\}) \).

First consider the case that \( i^* \) clinches an unlurked object \( y \in \bar{X}^\ell(h^*) \). If \( y \notin C_j^\subseteq(h^*) \), then the argument is exactly the same as in Case (1) (note that \( j \) is not ordered in step 1 in this case). If \( y \in C_j^\subseteq(h^*) \), then the step 1 partial order is \( \ell_1 \succ^1 \cdots \succ^1 \ell_n \succ^1 \{i^*, j\} \). We must show that any serial dictatorship run under \( f_\sigma : \ell_1, \ldots, \ell_n, i^*, j, \ldots \), and \( f'_\sigma : \ell_1, \ldots, \ell_n, j, i^*, \ldots \), result in the same outcome as \( \Gamma_\sigma \) for these agents. For the lurkers, the argument is as above in either case. For \( i^* \) and \( j \), in game \( \Gamma_\sigma \), by construction, \( y \in C_j(h') \) for some \( h' \subset h^* \). Let \( z = Top(\succ_j, \bar{X}^\ell(h^*)) \), and note that by Lemma 16, \( z \succ_j y \). Since \( i \) clinched \( y \) at \( h^* \), we have \( y \succ_i z \). In the serial dictatorship, after all lurkers have picked, the set of remaining objects is precisely \( \bar{X}^\ell(h^*) \). Thus, it does not matter whether \( i^* \) or \( j \) is ordered next in the serial dictatorship, as there is no conflict between them: in both cases, \( i^* \) will take \( y \), and \( j \) will take \( z \), and both \( f_\sigma \) and \( f'_\sigma \) give the same allocation as \( \Gamma_\sigma \). For the case where \( i^* \) begins

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\(^{85}\)For agent \( i^* \), since she took a lurked object at \( h^* \) in \( \Gamma_\sigma \), we have \( x_m = Top(\succ_i, X) \), and thus, at her turn in the serial dictatorship, she will once again select \( x_m \), since it is still available.

\(^{86}\)When it is agent \( \ell_m \)'s turn in the SD, the set of available objects is a subset of the set of objects that were offered to her when she clinched in \( \Gamma_\sigma : X \setminus \{x_1, \ldots, x_{m-1}\} \subseteq X \setminus \{x_1, \ldots, x_{m}\} \). However, \( x_{m'} \) belongs to both sets, and so since \( \ell_m \) took \( x_{m'} \) in \( \Gamma_\sigma \), she will also to take it at her turn in the SD, when her offer set is smaller.
by clinching some lurked object \(x_m \in \mathcal{X}^c(h^*)\), we consider agent \(j\) and the lurker who, in the chain of assignments, eventually takes an unlurked object \(y\); otherwise, the argument is analogous.

This shows that we get the same allocation for all agents ordered in step 1 of the ordering algorithm. If all active agents at \(\mathcal{A}(h^*)\) are processed in step 1 of the ordering algorithm, then we effectively have a smaller subgame on the remaining agents, and we just repeat the same argument. If not, then there is at most one active agent \(j \in \mathcal{A}(h^*)\) who is not processed in step 1. Agent \(j\) has been previously offered some objects in the set \(C_j^c(h^*)\) (note that \(C_j^c(h^*) \subseteq \mathcal{X}^c(h)\)). The subgame that begins after all of the agents in step 1 have clinched can equivalently be written as a Pareto efficient millipede subgame that begins with agent \(j\) being offered \(C_j^c(h^*)\) at the “root node”. We then find the first agent to clinch (after a series of passes) in this subgame, and repeat the same argument as for step 1 above.

**Proof of Lemma 8**

We will first show the result for \(k = 1\), and the proof for remaining steps will follow recursively. We first show that \(h_A^1 = h_B^1\). Towards a contradiction, assume \(h_A^1 \neq h_B^1\), and, wlog, \(h_A^1 \subsetneq h_B^1\). Let \(r = \rho(h_A^1)\) be the role associated with history \(h_A^1\), and define \(i_A = \sigma_A(r)\) and \(i_B = \sigma_B(r)\), where \(i_A \neq i_B\). Since there can be at most one passing action at \(h_A^1\) and \(h_A^1 \subsetneq h_B^1\), agent \(i_A\) must clinch some \(x_A\) at \(h_A^1\) in \(\Gamma_A\), while agent \(i_B\) must pass at \(h_A^1\) in \(\Gamma_B\).

Let \(\mathcal{A}_R(h_A^1)\) be the set of active roles at \(h_A^1\). Note that by definition, \(\mathcal{L}_R(h_A^1) \subseteq \mathcal{L}_R(h_B^1)\) and \(\mathcal{X}^c(h_A^1) \subseteq \mathcal{X}^c(h_B^1)\). Also, for the constructed partial order \(\triangleright_A\), let \(g_A(i) = |j : j \triangleright_A i| + 1\).

Define \(g_B\) similarly. Since we assume \(\triangleright_A^1 = \triangleright_B^1\), we have \(g_A(i) = g_B(i)\) for all \(i\) ordered in step 1 of the ordering algorithm applied to \(\Gamma_A\) and \(\Gamma_B\), respectively.

Since there is a passing action at \(h_A^1\), we have \(x_A \notin \mathcal{X}^c(h_A^1)\) (by BG Lemma 11). Since \(i_A\) clinches an unlurked object \(x_A \in \mathcal{X}^c(h_A^1)\) at \(h_A^1\), we have \(x_A = \text{Top}(\triangleright_A, \mathcal{X}^c(h_A^1))\), and also \(g_A(i_A) = \lambda(h_A^1) + 1\), where \(\lambda(h_A^1) = |\mathcal{L}_R(h_A^1)|\) is the number of lurkers that are present at \(h_A^1\). Therefore, \(g_B(i_A) = \lambda(h_A^1) + 1\) as well, which implies \(\mathcal{X}^c(h_A^1) = \mathcal{X}^c(h_B^1)\). This also means that \(\mathcal{L}_R(h_A^1) = \mathcal{L}_R(h_B^1)\). Let \(x_B\) be the object clinched at \(h_B^1\).

**Case (1):** \(x_B \notin \mathcal{X}^c(h_B^1)\).

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\(^{87}\)This is almost the same as \(i\)'s picking order in the resulting serial dictatorship, except that this allows for the fact that two agents may tie under \(\triangleright_A\), i.e., \(g_A(i) = g_A(i')\) if \(i \not\sim_A i'\) and \(i' \not\sim_A i\).

\(^{88}\)This follows because \(\mathcal{X}^c(h_A^1) \subseteq P_{\sim_A}(h_A^1) \cup C_{\sim_A}^c(h_A^1)\), by Lemma 10.

\(^{89}\)To see this, note that if \(\mathcal{X}^c(h_B^1) \supseteq \mathcal{X}^c(h_A^1)\), then \(x_{\lambda(h_A^1)+1} = x_A\), i.e., the \((\lambda(h_A^1) + 1)^{th}\) lurked object must be \(x_A\) (because the ordering algorithm puts the agent who receives \(x_{\lambda(h_A^1)+1}\) as the \((\lambda(h_A^1) + 1)^{th}\) agent in the ordering, and we know \(g_B(i_A) = \lambda(h_A^1) + 1\)). Thus, let \(h' \supseteq h_A^1\) be the history where \(x_A\) first becomes lurked. Note that \(x_A \in C_{\sim_A}^c(h')\). However, role \(r\) is still a non-lurker at \(h'\), and so \(x_A\) cannot become lurked (see the definition of lurker / Remark 2).
Subcase (1).(i): $\rho(h_B^1) \neq r$. There can be at most one other active non-lurker role at $h_B^1$, denoted $s \in \mathcal{L}_R(h_B^1)$. We have $\sigma_B^{-1}(r) \neq i_A$ (or else $i_A$ would again clinch $x_A$ at $h_A^1$), and $\sigma_B^{-1}(i_A) \neq r_n$ for any lurker role $r_n \in \mathcal{L}_R(h_B^1)$ (because then $g_B(i_A) = n' < \lambda(h_A^1) + 1$, a contradiction).\(^{90}\) Thus, it must be that $\sigma_B(s) = i_A$, $i_{h_B^1} = i_A$, and $i_A$ clinches $x_A$ at $h_B^1$ (i.e., $x_B = x_A$). By construction of $\tilde{v}_B^1$, $x_A \in C_{\tilde{v}_B^1}(h_B^1)$, where $j = \sigma_B(r)$, and so $g_B(i_A) = g_B(j) = \lambda(h_B^1) + 1 = \lambda(h_A^1) + 1$. So, $g_A(j) = \lambda(h_A^1) + 1$; in other words, in $\Gamma_A$, when $i_A$ clinches $x_A$ at $h_A^1$, $j$ must be an active non-lurker at $h_A^1$, and $x_A \in C_{\tilde{v}_B^1}(h_A^1)$, where $s = \sigma_A^{-1}(j)$ is $j'$s role in $\Gamma_A$. Since $\sigma_B(s) = i_A$, in game $\Gamma_B$, there is some $h' \subset h_A^1$ such that $x_A \in C_{i_A}(h')$ and $i$ passes at $h'$. Let $\bar{x} = \text{Top}(\succ_{i_A}, \mathcal{X}(h_A^1))$, and note that by Lemma 16, $\bar{x} \succ_{i_A} x_A$. However, we saw above that $x_A = \text{Top}(\succ_{i_A}, \mathcal{X}(h_A^1))$, which is a contradiction.

Subcase (1).(ii): $\rho(h_B^1) = r$. In game $\Gamma_B$, $i_{h_B^1} = i_B$, and, at $h_B^1$, $i_B$ clinches some $x_B \neq x_A$. Since $x_B$ is unlurked at $h_B^1$, we have $x_B = \text{Top}(\succ_{i_B}, \mathcal{X}(h_B^1))$. Since $g_B(i_A) = \lambda(h_A^1) + 1$ (as required by $\tilde{v}_B^1 = \tilde{v}_B^1$), we have $i_A \in \mathcal{A}(h_B^1)$ and $x_B \in C_{\tilde{v}_B^1}(h_B^1)$ in $\Gamma_B$. This again implies $g_A(i_A) = g_A(i_B) = g_B(i_A) = g_B(i_B) = \lambda(h_A^1) + 1$, i.e., $i_A$ and $i_B$ tie in both $\tilde{v}_A^1$ and $\tilde{v}_B^1$. Let $s = \sigma_B^{-1}(i_A)$. Since $g_A(i_B) = \lambda(h_A^1) + 1$, $i_B$ must have been an active non-lurker at $h_A^1$ in $\Gamma_A$, which means that $\sigma_B(i_B) = s$. Therefore, in game $\Gamma_A$, $i_B$ passes at some history $h' \subset h_A^1$ such that $x_B \in C_s(h')$. An argument equivalent to the previous paragraph applied to $i_B$ again reaches a contradiction.

Case (2): $x_B \in \mathcal{X}(h_B^1)$. Note that in this case, since a lurked object is clinchable at $h_B^1$, there is no passing action at $h_B^1$, by BG Lemma 11. Further, the role/agent who moves at $h_B^1$ satisfies the conditions of the terminator $t$ defined in BG Lemma 14; denote $\rho(h_B^1) = t$, and note that $C_{t}(h_B^1) = \mathcal{X}(h_B^1) = \mathcal{X}$. Also, recall from the discussion before Case (1) that $g_A(i_A) = \lambda(h_A^1) + 1$, where $\lambda(h_A^1) = |\mathcal{L}_R(h_A^1)|$, and therefore, $g_B(i_A) = \lambda(h_A^1) + 1$ as well.

Subcase (2).(i): $t = r$. In this case, in $\Gamma_A$, when $i_A$ clinches $x_A$ at $h_A^1$, we have $x_A = \text{Top}(\succ_{i_A}, \mathcal{X})$ (because $\sigma_A^{-1}(i_A) = t$ and $i_A$ chose to clinch first). Now, since $\sigma_B^{-1}(i_A) \neq t$, the only way for $g_B(i_A) = \lambda(h_A^1) + 1$ is for $i_A$ to be the active non-lurker at $h_B^1$ that does not clinch, and $y \in C_{i_A}(h_B^1)$, where $y$ is the unlurked object chosen by some lurker $i_t \in \mathcal{L}(h_B^1)$ in the assignment chain initiated when $i_B$ selected $x_B$ at $h_B^1$. Let $s = \sigma_B^{-1}(i_A)$. Since $i_t \in \mathcal{L}(h_B^1)$, and chose $y$ at her turn, we have $y = \text{Top}(\succ_{i_t}, \mathcal{X}(h_B^1))$. Note that $g_B(i_t) = \lambda(h_A^1) + 1$, and so, since $\tilde{v}_A^1 = \tilde{v}_B^1$, we have $g_A(i_t) = \lambda(h_A^1) + 1$ as well. This is only possible if $\sigma_A^{-1}(i_t) = s$, and $x_A \in C_{\tilde{v}_B^1}(h_A^1)$. But then, in game $\Gamma_B$, agent $i_A(= \sigma_B(s))$ was offered to clinch $x_A$ at some history $h' \subset h_B^1$. Since $x_A = \text{Top}(\succ_{i_A}, \mathcal{X})$, $i_A$ clinches at $h'$ in $\Gamma_B$, which is a contradiction.

Subcase (2).(ii): $t \neq r$. In this case, in $\Gamma_A$, when $i_A$ clinches $x_A$ at $h_A^1$, since $\sigma_A(t) \neq i_A$, we have $x_A \notin C_A(h_A^1)$, by BG Lemma 14. Therefore, $g_A(i_A) = \lambda(h_A^1) + 1$, and $g_A(i') \neq \lambda(h_A^1) + 1$ for all other $i' \neq i_A$ ordered in step 1 (in other words, $i_A$ does not tie with

\(^{90}\)Note that in this case, $x_B$ is unlurked, and so all lurkers are immediately assigned to their lurked objects.
another agent in $\bar{v}^1_A$), and so the same is true for $g_B$.\footnote{Note that since $\mathcal{L}_R(h^1_A) = \mathcal{L}_R(h^1_B)$, there cannot be any additional role $r' \notin \mathcal{L}_R(h^1_A) \cup \{r, t\}$ that is active at $h^1_A$.} Since the first agent to clinch in $\Gamma_B$ is $\sigma_B(t) = j \neq i_A$ who clinches some lurked object $x_B \in \mathcal{X}^L(h^1_B)$ and $\sigma_B(r) = i_B \neq i_A$, $\sigma_B^{-1}(i_A) = r_n$ for some lurker role $r_n$ that lurks object $x_n \in \mathcal{X}^L(h^1_B)$. Now, when $i_A$ eventually clinches $x_A$ (after someone else has selected $x_n$ in the chain of lurker assignments), $x_A \in C_{iB}^L(h^1_B)$, where $i_B = \sigma_B(r)$ (since $h^1_B \supseteq h^1_A$ and $x_A \in C_r(h^1_A)$), which implies that $g_B(i_B) = \lambda(h^1_A) + 1$, i.e., $i_B$ ties with $i_A$—a contradiction.

Thus far, we have shown that $\bar{v}^1_A = \bar{v}^1_B$ implies $h^1_A = h^1_B$, or, in other words, if step 1 of the ordering algorithm produces the same ordering, then step 1 must be initiated at the same history in $\Gamma_A$ and $\Gamma_B$. Next, we show that $\sigma_A(r') = \sigma_B(r')$ for all $r'$ that are ordered in step 1 of $\Gamma_A$ and $\Gamma_B$.

Define $h^1 := h^1_A = h^1_B$. Let $\mathcal{L}_R(h^1) = \{r_1, \ldots, r_{\lambda(h)}\}$ be the set of lurker-roles at $h^1$, and $\mathcal{X}^L(h^1) = \{x_1, \ldots, x_{\lambda(h)}\}$ the set of lurked objects. Notice that, since $h^1_A = h^1_B$, the lurked objects and active lurker-roles are equivalent in both $\Gamma_A$ and $\Gamma_B$. Towards a contradiction, assume that $\sigma_A(r') \neq \sigma_B(r')$ for some $r'$ that is ordered in step 1. Letting $r_0 = \rho(h^1)$, write

$$\sigma_A(r_0) \rightarrow x_{a_1} \rightarrow \sigma_A(r_{a_1}) \rightarrow x_{a_2} \rightarrow \cdots \rightarrow \sigma_A(r_{a_m}) \rightarrow y_A$$

(1)

to represent the chain of clinching that is initiated in $\Gamma_A$ by agent $\sigma_A(r_0)$ at $h^1$: agent $\sigma_A(r_0)$ clinches some (possibly lurked) object $x_{a_1}$, the agent $\sigma_A(r_{a_1})$ who was lurking $x_{a_1}$, clinches lurked object $x_{a_2}$, etc., until eventually agent $\sigma_A(r_{a_m})$ ends the chain by clinching some unlurked object $y_A$\footnote{Recall that any lurked object that does not appear in the chain is assigned to its lurker. For example, if $a_1 < a_2$, then $x_{a_1}$ is assigned to $\sigma_A(r_{a_1})$ for all $a'$ such that $a_1 < a' < a_2$.}. Similarly, for $\Gamma_B$, write

$$\sigma_B(r_0) \rightarrow x_{b_1} \rightarrow \sigma_B(r_{b_1}) \rightarrow x_{b_2} \rightarrow \cdots \rightarrow \sigma_B(r_{b_m}) \rightarrow y_B.$$  (2)

We will show that chains (1) and (2) above are in fact equivalent: $\sigma_A(r_0) = \sigma_B(r_0)$ and $\sigma_A(r_{a_m}) = \sigma_B(r_{b_m})$ for all $m$. Since any lurked object $x_n \in \mathcal{X}^L(h^1)$ that does not appear in the above chain must be assigned to its lurker $\sigma_A(r_n)$, this will deliver the result.

First, note that if $x_{a_1} = x_{b_1}$ and $x_{a_1} \notin \mathcal{X}^L(h^1)$, then both (1) and (2) begin with the same agent taking the same (unlurked) object. Therefore, all lurkers are immediately assigned to their lurked objects. If there is another active non-lurker role $s \in \mathcal{L}_R(h^1)$, then the agent in role $s$ clinches his object favorite remaining object. In either case, it is clear that $\sigma_A(r') = \sigma_B(r')$ for all roles $r'$ ordered in step 1 of the ordering algorithm. Thus, assume that $x_{a_1} \neq x_{b_1}$, and therefore, $\sigma_A(r_0) \neq \sigma_B(r_0)$.\footnote{Note that since $\mathcal{L}_R(h^1_A) = \mathcal{L}_R(h^1_B)$, there cannot be any additional role $r' \notin \mathcal{L}_R(h^1_A) \cup \{r, t\}$ that is active at $h^1_A$.}
Claim 2. At least one of \( x_{a_1} \) or \( x_{b_1} \) is lurked at \( h^1 \); i.e., \( \mathcal{X}^L(h^1) \cap \{x_{a_1}, x_{b_1}\} \neq \emptyset \).

**Proof of claim.** Assume that \( x_{a_1}, x_{b_1} \notin \mathcal{X}^L(h^1) \). For shorthand, define \( \sigma_A(r_0) = i_A \), and \( \sigma_B(r_0) = i_B \). Then, since \( \bar{\delta}_A^1 = \bar{\delta}_B^1 \), we have \( g_A(i_A) = g_A(i_B) = g_B(i_A) = g_B(i_B) = \lambda(h^1) + 1 \), i.e., \( i_A \) and \( i_B \) tie under both \( \bar{\delta}_A \) and \( \bar{\delta}_B \). This means that there is another active non-lurker role \( s \in \mathcal{A}_R(h^1) \setminus (\mathcal{L}_R(h^1) \cup \{r_0\}) \), and \( \sigma_A(r_0) = \sigma_B(s) = i_A, \sigma_A(s) = \sigma_B(r_0) = i_B \). Further, \( x_{a_1}, x_{b_1} \in C_s^C(h^1) \).

If \( \mathcal{L}(h^1) = \emptyset \), then \( \mathcal{X} = P_{r_0}(h^1) \cup C_{r_0}^C(h^1) \), which implies that \( x_{a_1} = \text{Top}(\succ_{i_A}, \mathcal{X}) \) and \( x_{b_1} = \text{Top}(\succ_{i_B}, \mathcal{X}) \). But, this means that in \( \Gamma_A \), \( i_B \) will clinic \( x_{b_1} \) at some \( h' \subsetneq h^1 \) (in particular, the earliest \( h' \) such that \( x_{b_1} \in C_s(h') \)), a contradiction. Therefore, \( \mathcal{L}(h^1) \neq \emptyset \).

Now, by BG Lemma 10, \( x_{a_1} = \text{Top}(\succ_{i_A}, \mathcal{X}^L(h^1)) \) and \( x_{b_1} = \text{Top}(\succ_{i_B}, \mathcal{X}^L(h^1)) \). Since \( x_{b_1} \in C_s^C(h^1) \), there is some history \( h' \subsetneq h^1 \) such that in game \( \Gamma_A \), \( x_{b_1} \in C_{i_B}(h') \), and \( i_B \) passes at \( h' \). By Lemma 16, \( \text{Top}(\succ_{i_B}, \mathcal{X}^L(h^1)) \succ_{i_B} x_{b_1} \), which is a contradiction.

By the previous claim, at least one (possibly both) of \( x_{a_1} \) and \( x_{b_1} \) are lurked objects; wlog, assume that \( x_{a_1} \in \mathcal{X}^L(h^1) \). Since \( C_{r_0}(h^1) \) contains a lurked object, BG Lemma 14 implies \( r_0 = t \) is the terminator role. Consider agent \( \sigma_A(r_{a_M}) = i' \), and note that \( x_{a_M} \succ_{i'} y_A = \text{Top}(\succ_{i'}, \mathcal{X} \setminus \{x_1, \ldots, x_{a_M}\}) \) where \( x_{a_M} \) is the object \( i' \) lurks at \( h^1 \) in \( \Gamma_A \).

We first claim that \( \sigma_B^{-1}(i') = r_{a_M} \) and \( y_A = y_B \). To see this, first note that \( \sigma_B^{-1}(i') \neq r_{n''} \) for any \( n'' > a_M \). Indeed, if this were the case, this would imply that \( x_{n''} = \text{Top}(\succ_{i'}, \mathcal{X} \setminus \{x_1, \ldots, x_{n''-1}\}) \), where \( x_{n''} \) is the object lurked by role \( r_{n''} \). However, this contradicts \( y_A = \text{Top}(\succ_{i'}, \mathcal{X} \setminus \{x_1, \ldots, x_{a_M}\}) \). Next, we show that \( \sigma_B^{-1}(i') \neq r_{n''} \) for any \( n'' < a_M \) either. In game \( \Gamma_B \), when \( i' \) becomes a lurker for \( x_{a_M} \) at some \( h' \), he eventually must get no worse than his second-best choice from the set \( P_{i'}(h') = \mathcal{X} \setminus \{x_1, \ldots, x_{n''-1}\} \). Since \( x_{a_M} \in P_{i'}(h') \), we have \( x_{n''} \succ_{i'} x_{a_M} \succ_{i'} y_A \), and \( i' \) can do no worse than \( x_{a_M} \), which means he cannot end up with \( y_A \)—a contradiction. The final case to consider is \( \sigma_B^{-1}(i') = r' \) for some \( r' \in \mathcal{L}_R(h^1) \). We cannot have \( r' = r_0 \) (since \( r_0 \) is the terminator role, \( i' \) would then be able to clinic her top choice at \( h^1 \) in \( \Gamma_B \), and \( \text{Top}(\succ_{i'}, \mathcal{X}) \neq y_A \)). Thus, \( r' \) must be the (unique) other non-lurker role that is active at \( h^1 \): \( r' = \mathcal{A}_R(h^1) \setminus (\mathcal{L}_R(h^1) \cup \{r_0\}) \). Recall that, for this role, \( P_{r'}(h^1) = \mathcal{X}^L(h^1) \). Further, \( y_A \notin C_{r'}^C(h^1) \) (or else \( i' \) would have clinched \( y_A \) at some strict subhistory of \( h^1 \)), and \( y_B \in C_{r'}^C(h^1) \) (or else \( i' \) would not be ordered in step 1 of the ordering algorithm under \( \Gamma_B \)). However, the former implies that \( |j : g_A(j) = \lambda(h^1) + 1| = 1 \), while the latter implies \( |j : g_B(j) = \lambda(h^1) + 1| = 2 \), which contradicts \( \bar{\delta}_A^1 = \bar{\delta}_B^1 \). Therefore, \( \sigma_A^{-1}(i') = \sigma_B^{-1}(i') = r_{a_M} \). Finally, note that \( \sigma_A(r_{a_M}) = \sigma_B(r_{a_M}) \) further implies \( y_A = y_B \) and \( r_{a_M} = r_{b_M} \). Indeed, if not, then the final person to clinic in chain 2 is some \( \sigma_B(r_{b_M}) = j \neq i' \). However, agent \( i' \) is a lurker in \( \Gamma_B \) for \( x_{a_M} \) that was not previously taken by any other agent in chain 2, and thus, \( i' \) is assigned to \( x_{a_M} \), which is a contradiction.

Next, consider agent \( \sigma_A(r_{a_M-1}) = i' \) in chain 1, i.e., \( i' \) lurks \( x_{a_M-1} \) in \( \Gamma_A \) and even-
ually ends up with (lurked) object \(x_{a_M}\). By construction of the chain, \(x_{a_M} = \text{Top}(\succ_i, \mathcal{X}\{x_1, \ldots, x_{a_M-1}\})\). Similar to the previous paragraph, \(\sigma^{-1}_B(i') \neq r''\) for any \(n'' > a_M - 1\). Indeed, if this were true, then \(x_{n''} = \text{Top}(\succ_i, \mathcal{X}\{x_1, \ldots, x_{n''-1}\})\). If \(n'' < a_M\), \(x_{n''} \succ_i x_{a_M}\), which contradicts \(x_{a_M} = \text{Top}(\succ_i, \mathcal{X}\{x_1, \ldots, x_{a_M-1}\})\). If \(n'' > a_M\), then \(x_{a_M}\) is not possible for \(i'\) (BG Lemma 12), which also is a contradiction. Finally, \(n'' \neq a_M\), since we already have shown that \(\sigma_B(r_{a_M}) = \sigma_A(r_{a_M})\). Similarly, \(\sigma^{-1}_B(i') \neq r''\) for any \(n'' < a_M - 1\), either, since this would imply that \(x_{n''} \succ_i x_{a_M-1} \succ_i x_{a_M}\), and \(x_{a_M-1} \in P_i(h')\) at the history \(h'\) where \(i'\) became a lurker for \(x_{n''}\). Since \(i'\) cannot do any worse than his second-best choice from \(P_i(h')\), we have a contradiction. The last case to consider is \(\sigma^{-1}_B(i') = r'\) for \(r' = \mathcal{A}_R(h^1) \setminus (\mathcal{L}_R(h^1) \cup \{r_0\})\). But, for this role, \(P_i(h^1) = \tilde{X}^C(h^1)\), and thus, no lurked objects are possible for \(i'\), which is a contradiction. Therefore, \(\sigma_A(r_{a_M-1}) = \sigma_B(r_{a_M-1})\). As in the previous paragraph, this also implies that \(r_{a_M-1} = r_{b_M-1}\) and \(x_{a_M} = x_{b_M}\).

The same argument can be repeated to show that \(x_a = x_b\) and \(\sigma_A(r_a) = \sigma_B(r_b)\) for all \(m = 1, \ldots, M\). The final case to consider is role \(r_0\). Let \(\sigma_A(r_0) = i'\). Since \(i'\) starts the chain of assignments at \(h^1\) by taking some lurked object \(x_{a_1} \in \mathcal{X}^C(h^1)\), we have \(x_{a_1} = \text{Top}(\succ_i, \mathcal{X})\). Once again, we cannot have \(\sigma^{-1}_B(i') = r''\) for any \(n'' < a_1\), as this would imply that \(x_{n''} \succ_i x_{a_1}\), which is a contradiction. We also cannot have \(\sigma^{-1}_B(i') = r_a\), since we have already shown that \(\sigma_B(r_a) = \sigma_A(r_a)\). Further, we cannot have \(\sigma^{-1}_B(i') = r''\) for any \(n'' > a_1\), since \(x_{a_1}\) would not be possible for \(i'\). Last, we cannot have \(\sigma^{-1}_B(i') = r'\) for \(r' = \mathcal{A}_R(h^1) \setminus (\mathcal{L}_R(h^1) \cup \{r_0\})\), since no lurked objects are possible for the agent in role \(r'\). Therefore, \(\sigma^{-1}_B(i') = r_0\), and chains 1 and 2 are equivalent.

To summarize: We have shown that if we have two role assignment functions \(\sigma_A, \sigma_B\) such that \(\tilde{z}_A = \tilde{z}_B\), then (i) \(\sigma_A^{-1}(i) = \sigma_B^{-1}(i)\) for all agents \(i\) that are ordered in step 1 of the ordering algorithm and (ii) at the conclusion of the chain of clinching initiated by the first agent to start the chain at \(h^1_A/h^1_B\), we will end up at the same history in \(\Gamma_A\) as in \(\Gamma_B\). At this point, we have a smaller subgame consisting of the agents and objects that were unmatched in the first round. This subgame is another Pareto efficient millipede game (that may begin with the unique unmatched agent from step 1 carrying over his endowment \(C_j(h^1)\), if such an agent exists), and so we simply repeat the above arguments for each round \(k = 1, \ldots, K\).

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93The case where \(\sigma^{-1}_B(i') = r_0\) can be dispensed with similarly as in the previous paragraph.

94This follows because the chain of clinching starts at the same history in both games, and all agents who are ordered in step 1 are in the same roles, so will take the same actions. While there may be some other role \(r'\) that is not ordered in step 1 and \(\sigma_A(r') \neq \sigma_B(r')\), this agent must have passed every time she was called to move at a history \(h' \subseteq h^1\), and is not called to move in the chain of lurker assignments, and so at the end of the chain, we will still end up at the same history to begin the next round/subgame.
Proof of Lemma 9

Assume there are three permutations $\sigma_A$, $\sigma_B$, and $\sigma_C$ that deliver (initial) partial orderings $\triangleright_A, \triangleright_B, \triangleright_C$ as in the statement. We’ll show that these 3 conditions lead to a contradiction.

As with Lemma 8, we show this first for the case that under $\triangleright_A$, all agents $\{i_1, \ldots, i_n, i, j\}$ are processed in step 1 of the ordering algorithm, and the argument for later steps will be equivalent. Let $\Gamma_A$, $\Gamma_B$, and $\Gamma_C$ denote the specific games under role assignments $\sigma_A, \sigma_B$, and $\sigma_C$ respectively. Further, let $h^*_A, h^*_B,$ and $h^*_C$ be the first history at which an object is clinched in the respective games, following a sequence of passes.\(^{95}\) In particular, this means that in $\triangleright_A$, agents $\{i_1, \ldots, i_n\}$ are getting lurked objects $\mathcal{X}(h^*_A) = \{x_1, \ldots, x_n\}$, while agents $i$ and $j$ are getting some unlurked objects, $y, z \notin \mathcal{X}(h^*_A)$, respectively. By construction, one of $i$ or $j$ must be an active non-lurker at $h^*_A$ who is not called to move at $h^*_A$; without loss of generality, assume that this is $j$. For notational purposes, denote by $i_{h^*_A} \in \{i_1, \ldots, i_n, i\}$ the agent who moves (and clinches) at $h^*_A$, thereby starting the chain of lurker assignments that ends with $i$ clinching $y$ followed by $j$ clinching $z$ (note that $i_{h^*_A} \notin \mathcal{L}(h^*_A)$, by Lemma 15, and it is possible that $i_{h^*_A} = i$). The structure of $\triangleright_A$ implies that $y \in C^\mathcal{E}_y(h^*_A)$.

**Case 1: $\mathcal{L}(h^*_A) = \emptyset$**

In this case, by definition of $\triangleright_A$, the set of active agents at $h^*_A$ is $\mathcal{A}(h^*_A) = \{i, j\}$, where $i_{h^*_A} = i$. For notational purposes, let $s = \sigma^{-1}_A(i)$ and $s' = \sigma^{-1}_A(j)$ be the roles assigned to agents $i$ and $j$ in game $\Gamma_A$, and note that $y \in C^\mathcal{E}_y(h^*_A)$. Further, both $i$ and $j$ are getting their first choice objects, i.e., $\text{Top}(\succ_j, \mathcal{X}) = y$ and $\text{Top}(\succ_j, \mathcal{X}) = z$.

Now, consider $\sigma_B$, where the ordering algorithm produces $i \triangleright_B j$. By Remark 2, $y$ cannot be the first lurked object in the game, and thus, for $i$ to be ordered first without ties according to $\triangleright_B$, we must have $\mathcal{X}(h^*_B) = \emptyset$ and $i_{h^*_B} = i$. These facts imply that $\sigma^{-1}_B(i) = s'$.

Now, consider $\sigma_C$, which begins $j \triangleright_C i \cdots$. There are two subcases, depending on whether $j$ is a lurker at $h^*_C$, or not.

**Subcase 1.(i):** $j \in \mathcal{L}(h^*_C)$. In this case, by construction, $j$ is the first lurker of the game and $z$ is the first lurked object. Further, $h^*_C \supseteq h^*_A$. Now, in order for $i$ to be the (unique) next agent added to $\triangleright_C$, either (i) $y \in \mathcal{X}(h^*_C)$ and, in particular, $y$ is the second object to become lurked in the game or (ii) $\mathcal{X}(h^*_C) = \{z\}$ and $i$ clinches $y$ at $h^*_C$. But, by Remark 2, $y$ cannot be the second lurked object of the game, because $y$ was previously offered to both roles $s$ and $s'$, and, even after $z$ becomes lurked by $j$ at some history $h'$, we will still have $y \in C^\mathcal{E}_y(h')$ for the role $r' \in \{s, s'\}$ such that $\sigma^{-1}_C(j) \neq r'$. Therefore, $i$ must clinch $y$ at $h^*_C \supseteq h^*_A$. Now, we have $\sigma^{-1}_C(i) \neq s, s'$ (because $\text{Top}(\succ_i, \mathcal{X}) = y$, and so $i$ would have clinched $y$ earlier along the path to $h^*_C$, since it has been previously offered to both roles).

\(^{95}\)That is, $h^*_a = (h_0, a^*, \ldots, a^*)$ for $a = A, B, C$, though the number of passes ($a^*$s) may vary.
Therefore, agent \( k = \sigma_C(r') \) is an active non-lurker at \( h_C^* \), such that \( y \in C_k^C(h_C^*) \), and so, by construction of \( \triangleright_C \), we have \( j \triangleright_C \{ i, k \} \triangleright_C \cdots \), which is a contradiction.

**Subcase (ii):** \( j \notin L(h_C^*) \). In this case, \( L(h_C^*) = \emptyset \) and \( i_{h_C^*} = j \) (since \( j \) is ordered first without ties). Further, \( \sigma_{C^{-1}}(j) \in \{ s, s' \} \). If \( \sigma_{C^{-1}}(j) = s' \), then \( \sigma_C(s) = k \neq i \) (or else we are back to \( \sigma_A \)) and \( h_C^* \supset h_A^* \). Thus, \( y \in C_k^C(h_A^*) \), which implies that \( i \) cannot be the next agent added to \( \triangleright_C \) uniquely—a contradiction.\(^96\) The last case is \( \sigma_{C^{-1}}(j) = s \). Note that \( z \notin C_s^C(h_B^*) \),\(^97\) and so \( h_C^* \supset h_B^* \). This further implies that \( y \in C_s^C(h_C^*) \) and so \( \sigma_C(s') = k \neq i \) (since otherwise, \( i \) would have clinched \( y \) prior to \( h_C^* \), because \( Top(\succ i, X) = y \)). By an argument similar to footnote 96, \( i \) cannot be the next agent added to the order uniquely, which is again a contradiction.

This completes the argument for Case 1.

**Case 2:** \( L(h_A^*) \neq \emptyset \).

Now, we consider the case where there are lurkers at \( h_A^* \) (and hence also lurked objects, \( X^L(h_A^*) \neq \emptyset \)). By definition of \( \triangleright_A \), we have \( A_R(h_A^*) = \{ r_1, \ldots, r_n, s, s' \} \), where \( r_1, \ldots, r_n \in L_R(h_A^*) \) are lurker roles, and \( s, s' \in L_R(h_A^*) \) are non-lurker roles. Let \( \rho(h_A^*) = s \) be the non-lurker role that moves at \( h_A^* \) (and thus clinches in \( \Gamma_A \)).

**Claim 3.** At \( h_A^* \), \( y \) has been previously offered to both active non-lurker roles: \( y \in C_s^C(h_A^*) \) and \( y \in C_{s'}^C(h_A^*) \).

**Proof of claim.** If \( \sigma_A(s) = i \), then it is obvious that \( y \in C_s^C(h_A^*) \) by definition. If \( \sigma_A(s) = i_{n'} \), then at \( h_A^* \), agent \( i_{n'} \) clinches a lurked object \( x_{n'} \in C_s(h_A^*) \), which initiates the chain of lurker assignments.\(^98\) This implies that there is no passing action at \( h_A^* \) (by Lemma 11) and that \( s \) is the terminator role \( t \) defined in BG Lemma 14. Therefore, by BG Lemma 14, \( y \in C_s^C(h_A^*) \). That \( y \in C_{s'}^C(h_A^*) \) for the other non-lurker role \( s' \) follows immediately from the construction of \( \triangleright_A \).

**Claim 4.** At \( h_B^* \), \( X^L(h_A^*) \subset X^L(h_B^*) \). Similarly, at \( h_C^* \), \( X^L(h_A^*) \subset X^L(h_C^*) \).

**Proof of claim.** Assume not, and let \( X^L(h_B^*) = \{ x_1, \ldots, x_{n'} \} \subset X^L(h_A^*) = X^L(h_B^*) \cup \{ x_{n'+1}, \ldots, x_n \} \). (Recall that following a sequence of passes to start the game, there is a unique order in which objects will become lurked that is independent of the role assignment function. Since \( h_A^* \) and \( h_B^* \) are by definition the first histories at which an object is clinched

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\(^96\)For agent \( i \) to be ordered next in \( \triangleright_C \) without ties, she must be the first agent ordered in step 2 of the ordering algorithm. By Remark 2 again, \( y \) cannot be the next lurked object in the game, which means that \( i \) must clinch \( y \) at some \( h' \supset h_C^* \) such that \( L(h') = \emptyset \). But, \( y \in C_k^C(h') \) and so, by construction of \( \triangleright_C \), she will tie with agent \( k \).

\(^97\)If \( z \in C_s^C(h_B^*) \), then \( \sigma_{B^{-1}}(s) = k \neq j \) (if \( \sigma_{B^{-1}}(s) = j \), then \( j \) would clinch \( z \) prior to \( h_B^* \), since \( z = Top(\succ j, X) \)). But, by the same argument in footnote 96, \( j \) could not be the next agent added to \( \triangleright_B \) uniquely, a contradiction.

\(^98\)By definition, \( \sigma_A(s) \neq j \), and so this exhausts all possibilities.
in their respective games, at least one of $X^L(h_A^*) \subseteq X^L(h_B^*)$ or $X^L(h_B^*) \subseteq X^L(h_A^*)$ must hold). Agent $i_{h_B^*}$ (the agent who clinches at $h_B^*$) cannot clinch $x_\bar{n}$ for any $\bar{n} \leq n'$. To see why, note that this would imply that agent $i_{h_B^*}$ is offered a previously lurked object at $h_B^*$. By BG Lemma 11, there is no passing action at $h_B^*$, which contradicts $x_{n'+1} \in X^L(h_A^*)$.

Thus, the only other possibility consistent with $\triangleright_B$ is that $i_{h_B^*} = i_{n'+1}$, who clinches $x_{n'+1}$ (which is unlurked) at $h_B^*$, i.e., $x_{n'+1} \in C_{i_{n'+1}}(h_B^*)$. But, $x_{n'+1}$ is the (unique) next object to become lurked in game form $\Gamma$ following a sequence of passes from $h_B^*$, which implies that $x_{n'+1} \notin C_{i_{n'+1}}(h_B^*)$, by Remark 2. An analogous argument applies for $h_C^*$. $$\square$$

Claim 5. $X^L(h_A^*) = X^L(h_B^*) = X^L(h_C^*)$.

Proof of claim. Given Claim 4, it is sufficient to show $X^L(h_B^*) = X^L(h_C^*) \subseteq X^L(h_A^*)$. Let $X^L(h_A^*) = \{x_1, \ldots, x_n\}$, and note again that the order in which objects become lurked following a series of passes to start the game is unique and independent of the role assignment function. Thus, it is sufficient to consider the next object that can become lurked, $x_{n+1}$, and show that $x_{n+1} \notin X^L(h_B^*)$ (resp. $x_{n+1} \notin X^L(h_C^*)$). Thus, assume that $x_{n+1} \in X^L(h_B^*)$, and note that this implies $h_B^* \supseteq h_A^*$. By construction of $\triangleright_B$, we must have $n+1 = y$ (the object received by $i$); indeed, if $x_{n+1} \neq y$, then, the agent, say $k$, who receives $x_{n+1}$ will be such that $k \triangleright_B i$, which is a contradiction. However, since $h_B^* \supseteq h_A^*$, $y$ has previously been offered to both active non-lurkers at $h_B^*$ (from the construction of $\triangleright_A$). Thus, by Remark 2, $y$ cannot become the next object lurked, i.e., $x_{n+1} \neq y$—a contradiction.

Next consider $\Gamma_C$, and assume that $x_{n+1} \in X^L(h_C^*)$. Let $r_{n+1}$ be the role that lurks $x_{n+1}$, and $h_{n+1}$ be the history at which role $r$ becomes a lurker for $x_{n+1}$ (i.e., role $r_{n+1}$ passes at $h_{n+1}$, and becomes a lurker at $h' = (h_{n+1}, a^*)$). Note that $h' \supseteq h_A^*$. Further, from what we know about the structure of the game tree $\Gamma$ from $\sigma_A$, there is another active non-lurker role at $h_{n+1}$, denoted $\bar{r}$, and we have $y \in C^S_{r_{n+1}}(h_{n+1})$ and $y \in C^S_{\bar{r}}(h_{n+1})$. Now, since $i_1\triangleright_C \cdots \triangleright_C i_n\triangleright_C j\triangleright_C i \cdots$, it must be that $x_{n+1} = z$ (the object received by $j$). Since $i$ is uniquely ordered immediately after $j$, we have $i_{n+1} = j$, and $i_{h_C^*} = i$ who clinches $y$ at $h_C^*$. (The only other possibility is that there is another lurked object at $h_C^*$, $x_{n+2} = y$, but, by Remark 2, this is impossible, since $y$ has been previously offered to both active nonlurkers at $h_{n+1}$). Since $i$ is ordered uniquely, $\sigma_C(\bar{r}) = i$, by an argument equivalent to footnote 96. Now, this implies that $i$ was previously offered $y$ at some $h'' \subsetneq h_C^*$, and chose to pass, which implies $\text{Top}(\sim_i, P_i(h'')) = \bar{x} \succ_i y$. Letting $h''$ be the most recent subhistory such that $y \in C_i(h'')$ and $i$ passes, and noting that $i$ chose to clinch $y$ at $h_C^*$, we conclude that $\bar{x} \notin P_i(h_C^*)$ and $\bar{x} = z$. But, $z \notin X^L(h_A^*)$, which contradicts that she clinches $y$ in $\Gamma_A$. $$\square$$

\textsuperscript{99}Each time a new object becomes impossible for $i$ (due to becoming lurked by another agent), $i$ must once again be offered the opportunity to clinch $y$, by definition of a millipede game. Agent $i$ must have passed at all such opportunities (including $h''$) up to $h_C^*$, which implies that $\bar{x} = z$.

\textsuperscript{100}There are actually two subcases here: if $i_{h_A^*} \neq i$, then $i$ must be a lurker for some $x_{n'}$. At some point in
Claim 5 thus implies that $\mathcal{L}_R(h^*_A) = \mathcal{L}_R(h^*_B) = \mathcal{L}_R(h^*_C)$. Further, we know from BG Lemma 13 that there can be at most two active non-lurker roles at any history, which we will denote $s$ and $s'$. By construction, we know that both of these roles are in fact active at $h^*_A$, i.e., $\mathcal{A}_R(h^*_A) = \mathcal{L}_R(h^*_A) \cup \{s, s'\}$. For $h^*_B$, it is possible that only one of $s$ or $s'$ are active (but this can only occur if $h^*_B \subsetneq h^*_A$). A similar remark applies to $h^*_C$.

Claim 6. $h^*_B, h^*_C \subsetneq h^*_A$.

**Proof of claim.** First, assume $h^*_A \subseteq h^*_B$. Then, when agent $i$ clinches $y$ in $\Gamma_B$ (either at $h^*_B$, or in the chain of lurker assignments that follows), it has already been offered to both of the agents in roles $s$ and $s'$, including the (unique) active non-lurker that does not move at $i_{h^*_B}$, say agent $k$, and so $k$ is ordered in step 1 (and in particular, $k$ will “tie” with $i$), which is a contradiction to the definition of $\triangleright_B$.

Next, assume that $h^*_A \subseteq h^*_C$. The agents processed in step 1 of the ordering algorithm applied to $\Gamma_C$ are $\{i_1, \ldots, i_n, j\}$ (a set that does not include $i$), and the chain ends when $j$ clinches $z$. Since $h^*_C \supseteq h^*_A$, both $s, s' \in \mathcal{A}_R(h^*_C)$, and $y \in C^C_s(h^*_C)$ and $y \in C^C_s(h^*_C)$. Since $i$ is the next agent ordered in $\triangleright_C$ without ties (in step 2 of the ordering algorithm), we have $\sigma_C(s') = i$ and $\sigma_C(s) = j$ (by an argument equivalent to footnote 96). Since $i$ clinches $y \in \bar{\mathcal{X}}^E(h^*_A)$ in $\Gamma_A$, we have $y = \text{Top}(\triangleright_i, \bar{\mathcal{X}}^E(h^*_A))$. Since $\sigma_C(s') = i$, there is some $h' \subsetneq h^*_A$ such that $y \in C_i(h')$ and $i$ passes at $h'$ in $\Gamma_C$. By Lemma 16, $\text{Top}(\triangleright_i, \bar{\mathcal{X}}(h^*_A)) = \bar{x} \triangleright_i y$, which is a contradiction. ■

Claim 7. At $h^*_B$, we have $\mathcal{X}^E(h^*_B) \cap C_{r'}(h^*_B) = \emptyset$, where $\rho(h^*_B) = r'$. The same holds at $h^*_C$.

**Proof of claim.** By Claim 6, $h^*_B \subsetneq h^*_A$. This implies that there must be a passing action at $h^*_B$, which means that $x_{n'} \notin C_{r'}(h^*_B)$ for any $x_{n'} \in \mathcal{X}^E(h^*_B)$ by BG Lemma 11. An equivalent argument holds for $h^*_C$. ■

In words, Claim 7 says that the object that is clinched at $h^*_B/h^*_C$ is not lurked. Note that the claim also implies that in both $\Gamma_B$ and $\Gamma_C$, $\sigma_B(r_{n'}) = \sigma_C(r_{n'}) = i_{n'}$ for all $n' = 1, \ldots, n$, and so $i_{h^*_B} = i$ who clinches $y$ first in $\Gamma_B$, and $i_{h^*_C} = j$, who clinches $z$ first in $\Gamma_C$.

We can now finish the proof of Lemma 9. Recall that $s = \rho(h^*_A)$ is the role of the first agent to clinch in $\Gamma_A$, and $s'$ is the role of the other active non-lurker at $h^*_A$ (for $\Gamma_A$, we know that $\sigma_A(s') = j$). There are two cases.

**Case (1):** $\sigma_A(s) = i$. This is the case where the two non-lurkers at $h^*_A$ are $\{i, j\}$, so that $s = \sigma_A^{-1}(i)$ and $s' = \sigma_A^{-1}(j)$. Note that at $h^*_A$, both $s, s' \in \bar{\mathcal{L}}_R(h^*_A)$, and $y \in C^C_s(h^*_A)$. Now, consider $\sigma_B/\triangleright_B$. By the discussion following Claim 7, $i_{h^*_B} = i$. This implies that the lurker assignment chain, someone (either $i_{h^*_A}$, or an earlier lurker) takes $x_{n'}$; since $z$ is still unlurked at that point, it is possible for $i$. Similarly, if $i_{h^*_A} = i$, then $z$ is still unlurked at $h^*_A$, which again contradicts that $i$ clinches $y$.  

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\[ \sigma_B^{-1}(i) = s'. \]

Now consider \( \sigma_C/\succ_C \). Again, by the discussion following Claim 7, \( i_{h_C} = j \); further, \( \sigma_C^{-1}(j) = s. \) If \( z \in C_{s}(h_B^*) \), then, by an argument equivalent to footnote 96, it must be that \( \sigma_B(s) = j \). Further, \( h_B^* \nsubseteq h_C^* \) (otherwise, \( j \) would clinch at \( h_C^* \nsubseteq h_B^* \) in \( \Gamma_B \), since she has the same role in both games). Therefore, \( y \in C_{s}(h_C^*) \) (in particular, \( y \in C_{s}(h_B^*) \)). Again by the same argument as footnote 96, \( \sigma_C(s') = i \). But, \( \sigma_C(s') = i \) implies that \( i \) clinches at \( h_B^* \nsubseteq h_C^* \) in \( \Gamma_C \) (since \( i \) has the same role as in \( \Gamma_B \)), which is a contradiction. Finally, if \( z \notin C_{s}(h_B^*) \), we once again have \( h_B^* \nsubseteq h_C^* \) and so \( y \in C_{s}(h_C^*) \), and we reach the same contradiction as in the previous case.

**Case (2):** \( \sigma_A(s) \neq i \). In this case, in \( \Gamma_A \), \( i \) must be a lurker for some \( x_{n'} \), and the first agent to clinch is some \( i_{h_A} = i_{n_1} \) who clinches some lurked object \( x_{n_1} \). This causes a chain of assignments of lurked objects, that ends with some other agent \( i_{n'} \) taking \( x_{n'} \), after which \( i \) clinches \( y \) and all lurked objects \( x_{n'+1}, \ldots, x_n \) are immediately assigned to their lurkers. Note that \( y \in C_s(h_A^*) \) here, by construction of \( \Gamma_A \). So, we have \( \sigma_A^{-1}(s) = i_{h_A} \neq i \), and \( \sigma_A^{-1}(s') = j \). Since there is a lurked object \( x_{n'} \in C_s(h_A^*) \), there is no passing action at \( h_A^* \), by BG Lemma 11. Further, this implies that role \( s \) is the terminator role defined in BG Lemma 14.

In game \( \Gamma_B \), the discussion following Claim 7 again gives \( i_{h_B} = i \). If \( \sigma_B^{-1}(i) = s \), then, since \( i \) is the first agent to clinch and is the terminator, we have \( \text{Top}(\succ_i, X) = y \). But, in game \( \Gamma_A \), \( i \) was a lurker for some \( x_{n'} \neq y \) (since \( y \) is not a lurked object at \( h_B^* \)), which implies \( \text{Top}(\succ_i, X) \neq y \), a contradiction. Therefore, \( \sigma_B^{-1}(i) = s' \).

In game \( \Gamma_C \), the discussion following Claim 7 gives \( i_{h_C} = j \). Just as in Case (1) above, we can show that \( \sigma_C^{-1}(j) = s \). Since \( s \) is the terminator role of BG Lemma 14, and \( j \) clinches \( z \) at \( h_C^* \), we conclude that \( \text{Top}(\succ_j, X) = z \). If \( z \in C_s(h_B^*) \), then let \( h' \nsubseteq h_B^* \) be a history such that \( z \in C_s(h') \). By an argument equivalent to footnote 96, we have \( \sigma_B(s) = j \).

However, this implies that \( j \) must clinch \( z \) at \( h' \) in \( \Gamma_B \), which contradicts that \( i_{h_B} = i \). Thus, \( z \notin C_s(h_B^*) \), which implies that \( h_C^* \nsubseteq h_B^* \), and so \( y \in C_s(h_C^*) \) (in particular, \( y \in C'_s(h_B^*) \)). By an argument equivalent to footnote 96, \( \sigma_C(s') = i \). However, if \( \sigma_C(s') = i \), then \( i \) clinches at \( h_B^* \nsubseteq h_C^* \) in \( \Gamma_C \) (since \( \sigma_C(s') = i = \sigma_B(s') \)), which contradicts that the first agent to clinch in \( \Gamma_C \) is \( i_{h_C} = j \) at \( h_C^* \).

We have thus shown that there cannot be three (initial) partial orderings \( \succ_A, \succ_B, \succ_C \) of the form given in the statement of Lemma 9 for the first step of the ordering algorithm. For the remaining steps, notice that the subgame after clearing all of the agents in step 1

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101 In particular, we cannot have \( \sigma_B^{-1}(i) = s \) again, because this would imply \( h_B^* = h_A^* \), contradicting Claim 6.

102 If \( \sigma_C^{-1}(j) = s' \), then, \( j \) must pass at all \( h' \nsubseteq h_A^* \) at which she is called to play (since \( \sigma_A(s') = j \), and \( j \) passed at all such \( h' \) in \( \Gamma_A \)). Since \( i_{h_C} = j \), this implies \( h_C^* \nsubseteq h_A^* \), which contradicts Claim 6.
is is simply another millipede game with lurkers (possibly being with one agent from the first stage carrying over her “endowment” to the second). We then simply repeat the same arguments as above, step-by-step, until we reach the end of the game. This completes the proof of Lemma 9.

B.7 Proof of Theorem 10

Suppose a game Γ is strongly obviously strategy-proof, efficient, and satisfies equal treatment of equals. Our characterization of SOSP and efficient mechanisms tells us that we can assume that Γ′ is equivalent to an almost-sequential dictatorship, which can be run as follows: at each even-size history, including the empty history, Nature chooses an agent from among those agents who have yet to move, and this agent moves at the following odd-size history.\(^{103}\) If there are three or more objects or exactly one object still unallocated, then this agent selects his most preferred still available object and sends an additional message. If, for the first time, there are exactly two unallocated objects (and thus two agents yet to move), then the agent who moves either (i) selects his most preferred object and sends a message, or (ii) has a choice of clinching one of the two objects (and sending a message) or passing. In the latter case, Nature selects the other agent who has yet to move, who chooses his best object, and the agent who passed obtains the remaining object.

The reminder of the proof is by induction on the number \(k\) of agents that have been called to move. We will show that each time Nature calls an agent to move, it must select uniformly at random from all agents who have yet to be called. Consider \(k = 1\) (i.e., the first agent called), and suppose there are more than three agents and objects. Consider a preference profile where all agents rank objects in the same order, \(o_1 \succ o_2 \succ \cdots \succ o_N\). By equal treatment of equals, all agents must receive object \(o_1\) with equal probability. Since, at this profile, \(o_1\) is always taken by the first mover, we conclude that Nature must select each agent to be the first mover with equal probability.

Now, consider any \(k\)-th move, where \(k < N\), and assume that for all moves \(1, \ldots, k - 1\), each remaining agent at that point was called on by Nature with equal probability. Label the history under consideration as \(h\), and name the agents so that along the path to \(h\), the first mover is \(i_1\) who chooses \(o_1\), then \(i_2\) chooses \(o_2\), etc. until \(i_{k-1}\) chooses \(o_{k-1}\).

Suppose first that there are at least three objects left. We claim that each agent who has not moved yet has an equal chance to be called by Nature at history \(h\). To see why, consider some object \(o \neq o_1, \ldots, o_{k-1}\) and the preference profile in which each agent \(i_\ell\), for

\(^{103}\)We include this rephrasing as it originally helped us think about this problem, and hence we think that it might help the reader. It actually is not needed and essentially the same proof works for the standard almost-sequential dictatorship (millipede) construction.
\( \ell = 1, \ldots, k - 1 \), ranks objects so that

\[
o_1 \succ_i o_2 \succ_i \cdots \succ_i o_\ell \succ_i o \succ_i \cdots
\]

and other objects are ranked below \( o \). Let \( Y(h) = I - \{i_1, \ldots, i_{k-1}\} \) be the set of agents who have yet to move at \( h \), and assume that all \( i \in Y(h) \) have the same preferences and rank \( o \) first. The total probability that \( i \) receives \( o \) is equal to the sum of (1) the probability that Nature chooses \( i \) at \( h \) and (2) the probability that Nature chooses \( i \) to move at any other \( h' \neq h \) where \( o \) is still available. The crux of the argument is to note that for the preferences specified, for any other branch of the tree (that does not contain history \( h \)), object \( o \) will be claimed by someone at the \((k - 1)\)-th move or earlier. By the inductive hypothesis, each time Nature picks an agent to move at any of these histories, all agents in \( Y(h) \) have an equal probability of being picked, and, if they are, they will immediately claim object \( o \). This implies that (2) is the same for all agents \( i \in Y(h) \). Since the sum (1)+(2) must also be the same for all agents (by equal treatment of equals), we conclude that (1) is also the same for all \( i, j \in Y(h) \), i.e., the probability that \( i \) is chosen to move at \( h \) is equal to the probability that \( j \) is chosen to move at \( h \), as desired.

Last, suppose that there are exactly two objects \( o \) and \( o' \) left at history \( h \). With two objects, Bogomolnaia and Moulin (2001) show that Random Priority is the only strategy-proof and efficient mechanism that satisfies equal treatment of equals, and the claim follows from their work.\(^{104}\)

B.8 Proof of Theorem 11

Given any SOSP and efficient game \( \Gamma \), the proof of Theorem 8 shows how to construct an equivalent curated dictatorship \( \Gamma' \). By efficiency and the richness of the preference domain, all objects are possible for the first mover in \( \Gamma' \). Therefore, by definition of a curated dictatorship, the first mover is offered the opportunity to clinch any object immediately, so long as \(|O| \geq 3\). Similarly, efficiency and richness imply that the second mover is offered the opportunity to clinch any still-remaining object, so long as there are at least 3 or more objects remaining to be allocated. The same applies to the third, fourth, etc. movers until exactly two objects remain. When two objects, say \( \{x, y\} \), remain, efficiency still implies that both objects are possible for the penultimate mover. If this agent is offered the opportunity to

\(^{104}\)Bogomolnaia and Moulin actually prove that Random Priority is the only strategy-proof and ordinally efficient mechanism that satisfies equal treatment of equals when there are three objects and three agents, and with two objects Pareto efficiency and ordinal efficiency become equivalent. The two object case is much simpler than the three object problem that Bogomolnaia and Moulin study, and one can verify the above claim without reliance on Bogomolnaia and Moulin’s seminal analysis.
clinches either, the game is a serial dictatorship, which is a special case of an almost-sequential
dictatorship. Alternatively, he may be given a choice between clinching one of \( x \) or \( y \), say \( x \),
or passing and allowing the final mover to allocate \( x \) and \( y \) between them. In either case, it
is easy to see that the overall game \( \Gamma' \) is an almost-sequential dictatorship.

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