Matching with Externalities

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Abstract

We incorporate externalities into the stable matching theory of two-sided markets. Extending the classical substitutes condition to markets with externalities, we establish that stable matchings exist when agent choices satisfy substitutability. We show that substitutability is a necessary condition for the existence of a stable matching in a maximal-domain sense and provide a characterization of substitutable choice functions. In addition, we extend the standard insights of matching theory, like the existence of side-optimal stable matchings and the deferred acceptance algorithm, to settings with externalities even though the standard fixed-point techniques do not apply.

1 Introduction

Externalities are present in many two-sided markets. For instance, couples in a labor market pool their resources as do partners in legal or consulting partnerships. As a result, the preferences of an agent depend on the contracts signed by the partners. Likewise, a firm's hiring

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decisions are affected by how candidates compare to competitors' employees. Finally, because of technological requirements of interoperability, an agent's purchase decisions depend on other agents' decisions.[1]

In this paper, we incorporate externalities into the stable matching theory of Gale and Shapley (1962).[2] We refer to the two sides of the market as buyers and sellers. Each buyer-seller pair can sign many bilateral contracts. Furthermore, each agent is endowed with a choice function that selects a subset of contracts from any given set conditional on a reference set for the other agents. We build a theory of matching with externalities that both establishes new insights and extends to the settings with externalities some of the key insights of the classical theory without externalities, such as the existence of stable matchings and the role of the deferred acceptance (or cumulative offer) algorithm.[3]

Our theory is built on a substitutes condition that extends the classical substitutes condition to the setting with externalities. Our condition requires that each agent rejects more contracts from any set than its subsets conditional on the same reference set (as in the classical substitutes condition) and also that each agent rejects more contracts from a set X conditional on a reference set μ than set X conditional on a reference set μ' such that μ reflects better market conditions than μ' for her side of the market. The idea of better market condition extends the revealed preference idea of Blair (1988) to the setting with externalities. When there are no externalities, this substitutes condition reduces to the classical gross substitutes condition of Kelso and Crawford (1982). Our condition is satisfied by standard choice functions of households consisting of a primary and a secondary earner who pool resources; the pooling of resources implies that the choice function of a secondary earner depends on the income of the primary earner and hence exhibits externalities (see Section 3).

We first construct a version of the deferred acceptance algorithm that performs well despite the presence of externalities. This algorithm—which may be interpreted as a new ascending auction—may be useful in potential market-design applications. Because an agent's choice depends on others' contracts, our algorithm keeps track not only of which contracts are available

[1]These markets are discussed in more detail in Section 3 and Appendix F.

[2]Even though we derive our results in a general many-to-many matching setting with contracts (cf. Hatfield and Milgrom, 2005, Klaus and Walzl, 2009, and Hatfield and Kominers, 2017), the results are new in all special instances of our setting, including many-to-one and one-to-one matching problems.

[3]We focus on the classical short-sighted stability concept in which each agent assumes that other agents do not react to their choice. Our results, however, are applicable to many other stability concepts including far-sighted ones because we formulate the results in terms of agents' choice behavior and not in terms of their preferences. See Remark 1 of the previous version of our paper, which is available at `http://dx.doi.org/10.2139/ssrn.2475468`.

but also of the reference sets that agents on each side use to condition their choice. The construction requires care because after the reference set changes an agent may want to go back to a contract that is already rejected. To ensure that this does not happen, we construct the initial reference sets in a preliminary phase of the algorithm. Relatedly, we cannot stop the algorithm as soon as the sets of available contracts converge: we need to continue until the reference sets converge as well. Our construction of initial reference sets ensures that subsequent reference sets change in a monotonic way with respect to the better market conditions preorder, thus ensuring that from some point on the reference sets belong to the same equivalence class. While these equivalence classes might consist of many matchings, we further show that the algorithm converges to one of them and never cycles among the members of the same equivalence class.

Our main results show that there exists a stable matching when choice functions satisfy substitutability (Theorem 1) because the algorithm converges to one and that substitutability is necessary for the existence of a stable matching in a maximal-domain sense (Theorem 2) extending the insights of Hatfield and Milgrom (2005), Hatfield and Kojima (2010), and Hatfield and Kominers (2017) for the standard substitutability condition in settings without externalities.

In addition to the main results, we show that every stable matching is Pareto efficient (Theorem 3) and an optimal stable matching exists for side θ under the additional assumption that there exists a matching that reflects better market conditions than any other matching that can be chosen for side θ (Theorem 4). This additional assumption is satisfied trivially in settings without externalities, where the existence of side-optimal stable matchings was established by Gale and Shapley (1962) for the marriage problem. Furthermore, we provide a characterization of substitutable choice functions (Theorem 5): a choice function satisfies the substitutes condition if, and only if, the choice from a set consists of the highest ranked contracts according to some ranking, where the set of allowed rankings is fixed for the choice function. This characterization is inspired by the decomposition result of Aizerman and Malishevski (1981) for the setting without externalities.[4]

We also generalize the rural hospitals theorem (McVitie and Wilson, 1970; Roth, 1984; Hatfield and Milgrom, 2005), which states that each agent gets the same number of contracts in every stable matching in a many-to-one matching problem without externalities (in Appendix A). Our generalization allows different contracts to have different weights that may depend on the quantity, price, or quality of the contracts. For this purpose, we introduce a general

[4]For applications of such a decomposition result in settings without externalities see Chambers and Yenmez (2017).

law of aggregate demand. An agent's choice function satisfies *the law of aggregate demand* if the weight of contracts chosen from a set conditional on a reference set μ is greater than the weight of contracts chosen from a subset conditional on a reference set that has worse market conditions than μ. When there are no externalities, this law of aggregate demand reduces to the monotonicity condition of Fleiner (2003). We show that when choice functions satisfy the law of aggregate demand in addition to the aforementioned properties, all stable matchings have the same weight for every agent (Theorem 6).

Many of our results have no forerunners in the literature analyzing externalities in matching. These results include, to the best of our knowledge, our development of the substitutability condition (and its characterization) as well as our results on efficiency and side-optimal stable matchings.

The prior matching literature studying externalities focused on the question of existence of stable matchings and algorithms that can find them.[5] The subliterature analyzing the existence and nonexistence results largely builds on the seminal paper by Sasaki and Toda (1996), who showed that stable one-to-one matchings need not exist in the presence of externalities. They also proposed a weak stability concept that allows a pair of agents to block a matching only if they benefit from the block under all possible rematches of the remaining agents and showed that such weak stable matchings exist in one-to-one environments. Much of the subsequent literature—e.g., Chowdhury (2004); Hafalir (2008); Eriksson, Jansson and Vetander (2011); Chen (2013); Gudmundsson and Habis (2017); Salgado-Torres (2011*a*,*b*); Bodine-Baron et al. (2011)—maintained the focus on the existence question and proposed a variety of weak stability concepts that modify Sasaki and Toda's by varying the degree to which the rematches of other agents penalize the blocking pair. In contrast, our paper uses the standard stability concept of Gale and Shapley (1962) and the literature on matching without externalities.[6] We also contribute conceptually to this earlier literature by pointing out that agents' choice behavior—which in contrast to this literature we take to be a primitive of our modeling—synthesizes both agents' preferences and assumptions on other agents' reactions to a block.[7]

Our contribution on the existence question is closest to the few papers that look at standard stability in selected matching problems with externalities. Bando (2012; 2014) studies many-

[5]Bando, Kawasaki and Muto (2016) provide a recent survey.

[6]In this standard stability concept, a set of agents forms a blocking coalition if it benefits them in the absence of any reaction from the remaining agents. Note that even in the absence of externalities, one might entertain alternative solution concepts in which an agent might be unwilling to enter a blocking coalition if they are concerned that doing so will trigger a chain of events that will lead them to losing a partner they block with.

[7]See Footnote 3.

to-one matching allowing externalities in the choice behavior of firms (agents who match with potentially many agents on the other side) but not of workers; he further assumes that each firm's choice function depends on the matching of other firms only through the set of workers hired by other firms, and imposes several other assumptions on firms' choice behavior. Under these assumptions, he proves the existence of stable matchings and analyzes the deferred acceptance algorithm. In his setting there is no need to keep track of the reference sets in the deferred acceptance algorithm (and hence no need for the preliminary phase that constructs the initial reference sets), and his algorithm terminates as soon as there are no rejections. Our substitutes condition does not imply Bando's assumptions nor is implied by them. An advantage of our approach is that it is equally valid in one-to-one, many-to-one, and many-to-many matching settings, while Bando's conditions do not guarantee the existence of stable many-to-many matchings even when there are externalities only on one side of the market.[8]

Other analyses of the existence of matchings that are stable in the standard sense in settings with externalities focused on externalities within couples (Dutta and Massó, 1997; Klaus and Klijn, 2005; Kojima, Pathak and Roth, 2013; Ashlagi, Braverman and Hassidim, 2014) and on complementarities and peer effects among students matched to the same college or workers matched to the same firm (Dutta and Massó, 1997; Echenique and Yenmez, 2007; Pycia, 2012; Hatfield and Kominers, 2015).[9] not restricting our attention to either of these two types of externalities, we contribute to both of these subliteratures. Our model of couples in local labor markets—which is an example of our general framework—is complementary to these earlier analysis of externalities among couples; the externality of focus in these earlier analyses is that the members of the couple ending up with jobs that are far away rather than on the issue that a better job for one member of the couple might enable the other member to be more selective. In contrast, we focus on the latter issue. Our model of benchmarking in hiring—another example of our general framework—is complementary to the earlier analyses of complementarities; the key externality of focus in these earlier papers is one of production complementarities among agents matched with the same college or firm rather than on the issue of externalities across colleges or firms, such as those caused by benchmarking. We focus on the latter issue in our application of our general model to hiring.

Another important difference with the aforementioned papers is that they focused on suf-

[8] In Example 1, our substitutes condition is satisfied, Bando's assumptions are not, and a stable matching exists. In Example 2, our substitutes condition is not satisfied, Bando's assumptions are satisfied, and a stable matching does not exist.

[9] Ostrovsky (2008) studies complementarities in a supply chain network. Sun and Yang (2006) study complementarities in an exchange economy.

ficient conditions for existence, except for Pycia (2012) who also—like us—provided a corresponding necessity result. Within the confines of the college admission setting he studies, he showed that his preference alignment condition is not only sufficient but also necessary in a maximal-domain sense. Pycia's alignment condition is neither implied by nor implies standard substitutability as discussed in his paper; for the same reasons, his condition is neither implied by nor implies our condition.[10]

The second focus area of the previous literature that allowed externalities is algorithms that lead to stable matchings (Echenique and Yenmez, 2007; Pycia, 2012; İnal, 2015). These studies of the algorithmic question restricted attention to settings in which the complementarities and peer effects are only among market participants matched to the same agent on the other side of the market. The deferred acceptance algorithm we proposed is not restricted in this way.[11]

Our work is also related to the exploration of efficiency in markets with externalities (cf. Pigou (1932); Ashlagi and Shi (2014); Watson (2014); Chade and Eeckhout (2019)); while this literature focuses on efficiency, we focus on stability.[12] Another part of broader context is the literature on one-sided allocation that allows for substitutes and complementarities among assigned goods but usually assumes the absence of externalities across agents; cf. Budish (2011), Budish and Cantillon (2012), and Miralles and Pycia (2020). The main exception is Baccara et al. (2012), who analyze stable one-sided allocations and, in addition to an in-depth empirical analysis of office allocation at a university, they prove that stable one-sided allocations exist in the presence of externalities provided these externalities have no impact on agents' choice behavior; in contrast we allow externalities that may affect behavior.[13][14]

[10] In particular, his alignment condition fails in general in models with transfers because the receiver of the transfer prefers a higher payment while the sender prefers a lower payment (cf. Pycia, 2008). Mumcu and Saglam (2010) extend the alignment approach to analyze when all matchings in the non-empty collection of top matchings are stable and Teytelboym (2012) extends this approach to externalities among agents in a component of a network and shows that Pycia's alignment condition is then sufficient for the existence of a stable matching.

[11] On the other hand, our algorithm cannot substitute for the earlier proposals in their applicability settings. For instance, in the environment they study, Echenique and Yenmez (2007) constructed an algorithm that finds a stable matching whenever stable matchings exist.

[12] See also Uetake and Watanabe (2012) who provide an empirical analysis of firm mergers using a matching model with externalities.

[13] Hong and Park (2018) also study externalities that have no impact on agents' behavior in the context of house allocation; they assume that agents' preferences over objects do not exhibit externalities but allow agents to have lexicographically second order preferences over economy-wide assignments. Since mechanisms based on the top trading cycles algorithm are non-bossy, these second-order preferences have no impact on agents' behavior. Frys and Heller (2016) assume that agents are partitioned into groups of friends—any two friends have identical preferences and care about each other assignments, there are no externalities across friends—and study mechanisms based on the random serial dictatorship.

[14] As we consider an application of our results to the analysis of dynamic matching in Appendix F, let us observe that prior analyses of dynamic matching—e.g., Ünver (2010), Kurino (2014), and Kotowski (2015)—focused on

2 Model

There is a finite set of agents $\mathcal{I}$ partitioned into buyers, $\mathcal{B}$, and sellers, $\mathcal{S}$, $\mathcal{B} \cup \mathcal{S} = \mathcal{I}$. The set of agents on the same side with agent i is denoted as $\theta(i)$. Therefore, $\theta(i) = \mathcal{B}$ if i is a buyer and $\theta(i) = \mathcal{S}$ if i is a seller. With a slight abuse of notation, θ also denotes one side of the market, so $\theta \in \{\mathcal{B}, \mathcal{S}\}$. If θ is a side, then $-\theta$ is the other side, that is, $-\mathcal{B} \equiv \mathcal{S}$ and $-\mathcal{S} \equiv \mathcal{B}$. Agents interact with each other bilaterally through contracts. Each contract x specifies a buyer $b(x)$, a seller $s(x)$, and terms, which may specify price, quantity, and quality. There exists a finite set of contracts $\mathcal{X}$. For any $X \subseteq \mathcal{X}$, X_i denotes the set of contracts in X involving agent i, that is $X_i \equiv \{x \in X : i \in \{b(x), s(x)\}\}$. Similarly, X_{-i} denotes the set of contracts not involving agent i, that is, $X_{-i} \equiv X \setminus X_i$.

Each agent i has a choice function c_i, where $c_i(X_i|\mu_{-i})$ is the set of contracts that i chooses from a set X_i conditional on a reference set μ_{-i}, which is the set of contracts signed by the other agents on the same side.[15] The presence of externalities means that agents' choices are conditional on the state of the market, and to allow the conditioning, the state of the market should be observable by the agents. A natural observable is the matching that prevails on the market; and hence we condition the choices on the reference matching.

We expand the domain of the choice function so that $c_i(X|\mu) = c_i(X_i|\mu_{-i})$. Choice function c_i has externalities if there exist $X, \mu, \mu' \subseteq \mathcal{X}$ such that $c_i(X|\mu) \neq c_i(X|\mu')$; otherwise, the choice function exhibits no externalities. Let $r_i(X|\mu) \equiv X_i \setminus c_i(X|\mu)$ be the set of contracts rejected by agent i from X conditional on a reference set μ. Similarly, define $C^\theta(X|\mu) \equiv \cup_{i\in\theta} c_i(X|\mu)$ to be the set of chosen contracts and $R^\theta(X|\mu) \equiv \cup_{i\in\theta} r_i(X|\mu)$ to be the set of rejected contracts from set X by side θ conditional on a reference set μ. Note that for any $X, \mu \subseteq \mathcal{X}$ and side θ, $C^\theta(X|\mu)$ and $R^\theta(X|\mu)$ form a partition of X since every contract involves exactly one agent from each side of the market and is either accepted or rejected by the agent. A **matching problem** is a tuple $(\mathcal{B}, \mathcal{S}, \mathcal{X}, C^{\mathcal{B}}, C^{\mathcal{S}})$.

We use the term **matching** to refer to any set of contracts. We embed any quota constraints, if they exist, in agents' choice behavior. For instance, we model one-to-one matching markets by assuming that each agent chooses at most one contract from any set of contracts. Thus, examples of our setting include standard one-to-one and many-to-one matching problems with and without transfers.[16]

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environments without externalities; an exception is Pycia (2012), discussed above.

[15] We could allow choice functions c_i to depend not only on X_i and μ_{-i} but also on μ_i (that is the set of contracts signed by i) with no change in our proofs. A sole exception is the comment provided in Footnote 20.

[16] Without affecting any of the results, we could alternatively model one-to-one matching and other matching

A matching μ is **individually rational** for agent i if $c_i(\mu_i|\mu_{-i}) = \mu_i$. Less formally, conditional on the contracts of other agents on the same side, agent i wants to keep all of their contracts. A buyer i and seller j form a **blocking pair** for matching μ if there exists a contract $x \in \mathcal{X}_i \cap \mathcal{X}_j$ such that $x \notin \mu$ and $x \in c_i(\mu \cup \{x\}\,|\mu) \cap c_j(\mu \cup \{x\}\,|\mu)$. In words, a pair can block a matching μ if they both would like to sign a new contract conditional on μ. Matching μ is **stable** if it is individually rational for all agents and there are no blocking pairs. This stability concept is identical to pairwise stability studied in settings without externalities (Gale and Shapley, 1962). As in the standard settings without externalities, stability defined in terms of individual and pairwise blocking is equivalent to group stability when choice rules are substitutable; see Appendix B.

2.1 Properties of Choice Functions

To guarantee the existence of stable matchings, we impose more structure on the choice functions. First, we generalize two standard assumptions studied in the matching literature without externalities to our setting. Then, we introduce a new assumption, which is trivially satisfied when there are no externalities.

The first assumption is a basic rationality axiom we assume throughout the paper.

Definition 1. Choice function c_i satisfies the **irrelevance of rejected contracts** if for all $X_i, X_i' \subseteq \mathcal{X}_i$ and $\mu_{-i} \subseteq \mathcal{X}_{-i}$, we have

$$c_i(X_i'|\mu_{-i}) \subseteq X_i \subseteq X_i' \implies c_i\left(X_i|\mu_{-i}\right) = c_i(X_i'|\mu_{-i}).$$

If choice function c_i satisfies the irrelevance of rejected contracts, then excluding contracts that are not chosen does not change the chosen set.[17] This is a basic property of choice functions. It is equivalent to the *weak axiom of revealed preference* in settings without externalities (Alva, 2018). The irrelevance of rejected contracts has been recognized as an important property in the choice-function approach to matching by, e.g., Blair (1988) and Aygün and Sönmez (2013), who restricted attention to the case without externalities. The irrelevance of rejected contracts is satisfied in all our examples.

environments with quota constraints by assuming that only some sets of contracts are feasible matchings. This alternative route is straightforward if agents condition their choice behavior on any sets of contracts rather than on feasible matchings. As is usual in models of matching with contracts, in applications with transfers, we assume that there is a lowest monetary unit.

[17] All our assumptions on individual choice functions can equivalently be stated in terms of the side choice functions.

The second assumption rules out complementarities between contracts of an agent.

Definition 2. Choice function c_i satisfies **standard substitutability** if for all $X_i, X_i' \subseteq \mathcal{X}_i$ and $\mu_{-i} \subseteq \mathcal{X}_{-i}$,

$$X_i' \supseteq X_i \implies r_i\left(X_i'|\mu_{-i}\right) \supseteq r_i\left(X_i|\mu_{-i}\right).$$

A choice function satisfies standard substitutability if the corresponding rejection function is monotone for a fixed reference set, or equivalently, a contract that is chosen from a set is also chosen from any subset including that contract conditional on the same reference set. When there are no externalities, the choice behavior does not depend on the reference set and this assumption reduces to the condition introduced by Kelso and Crawford (1982) for a matching market with transfers.[18]

Our third assumption captures the idea that not only a single agent's contracts are substitutable but also a similar substitutability of contracts obtains across agents on the same side of the market. Roughly speaking, the intuition is that when all agents on one side of the market choose from larger sets, then each agent on this side rejects more contracts. We capture this intuition by imposing monotonicity of rejections in terms of a ranking on the reference matching.

To formalize the third assumption, we need the following definitions. A **binary relation** $\succsim_i$ on a domain $\mathcal{A}_i \subseteq 2^{\mathcal{X}_i}$ is a set of ordered pairs of matchings in $\mathcal{A}_i$; it is **reflexive** if for any $\mu_i \in \mathcal{A}_i$, $\mu_i \succsim_i \mu_i$; it is **transitive** if $\mu_i^1 \succsim_i \mu_i^2$ and $\mu_i^2 \succsim_i \mu_i^3$ imply $\mu_i^1 \succsim_i \mu_i^3$. A **preorder** is a reflexive and transitive binary relation. We restrict our attention to preorders $\succsim_i$ that have the empty set in their domain, so $\varnothing \in \mathcal{A}_i$.[19] Given a preorder $\succsim_i$ on a domain $\mathcal{A}_i \subseteq 2^{\mathcal{X}_i}$ for each agent i on side θ, we define the corresponding preorder $\succsim^\theta$ for side θ on domain $\mathcal{A} = \{\mu \subseteq \mathcal{X} : \mu_i \in \mathcal{A}_i\} \subseteq 2^{\mathcal{X}}$ as follows: for every $\mu, \mu' \in \mathcal{A}$,

$$\mu' \succsim^\theta \mu \iff \mu_i' \succsim_i \mu_i \; \forall i \in \theta.$$

In line with our motivation, when $\mu' \succsim^\theta \mu$ we say that μ' reflects **better market conditions** than μ for side θ. Using preorders of individual agents, a similar preorder $\succsim^{\theta'}$ can be defined for any set of agents $\theta' \subseteq \theta$.

An example of a preorder is the **revealed-preference order**, defined for the case when

[18] See also Roth (1984), Fleiner (2003), and Hatfield and Milgrom (2005). The substitutes condition is behind the monotonicity properties of the deferred acceptance algorithm when there are no externalities, and in this way underpins the standard matching analysis.

[19] Instead of preorders we can also work with a transitive binary relation satisfying $\varnothing \succsim_i \varnothing$.

choice functions do not have externalities: $\mu_i' \succsim_i \mu_i$ if, and only if, $c_i(\mu_i' \cup \mu_i) = \mu_i'$. In the matching context this revealed-preference order was introduced by Blair (1988), and hence it is sometimes called Blair order (Echenique and Oviedo, 2006). In general, not all matchings can be compared using the revealed-preference order and the comparison is reflexive only on the set of the fixed points of the choice function, $\{\mu_i \subseteq \mathcal{X}_i : c_i(\mu_i) = \mu_i\}$. Likewise, in our general case, if a matching μ_i is not in the domain $\mathcal{A}_i \subseteq 2^{\mathcal{X}_i}$ of the better market condition preorder, we cannot compare it to any other matching. While in Blair's setting the revealed-preference order is a partial order, that is an *antisymmetric* preorder, where antisymmetry means that no two distinct matchings can be related in both directions, our analysis requires us to use the more general concept of a preorder because antisymmetry might fail in the presence of externalities (cf. Example 1). In particular, an agent's choice from a given set of contracts may depend on the reference matching when there are externalities and as a result the better market condition may change depending on the reference set.

As in the Blair order, we only need to compare matchings that can be chosen. When the choice is conditional on the same reference matching, we need to be able to compare the matching chosen from a set X with any matching chosen from a subset of X. When the choice is conditional on different reference matchings, we need to be able to make comparisons implied by the following consistency assumption. A preorder $\succsim^\theta$ for side θ is **consistent** with the side choice function C^θ if, for any $i \in \theta$ and $X, X', \mu, \mu' \subseteq \mathcal{X}$,

$$X_i' \supseteq X_i \text{ and } \mu_{-i}' \succsim^{\theta\setminus\{i\}} \mu_{-i} \implies c_i\left(X_i' | \mu_{-i}'\right) \succsim_i c_i\left(X_i | \mu_{-i}\right).$$

Thus, if an agent chooses from a larger set and if the other agents have a better market condition, then the agent also has a better market condition. As in the revealed-preference order, when there are more alternatives to choose from the choice made reflects a better market condition than the choice made from fewer alternatives when the choice is conditional on the same reference matching. In addition, the same comparison holds when the choice from the superset is conditional on a reference matching that has a better market condition than the reference matching of the choice from the subset.

For every side choice function, there exists a preorder that is consistent. For example, the preorder that compares every pair of matchings is consistent. For the rest of the paper, we fix an arbitrary consistent preorder $\succsim^\theta$ unless otherwise stated.

We are now ready to state our third, and main, assumption.

Definition 3. Choice function C^θ satisfies **monotone externalities** if for all $i \in \theta$, $X_i \subseteq \mathcal{X}_i$, and

$\mu_{-i}, \mu'_{-i} \subseteq \mathcal{X}_{-i}$,

$$\mu'_{-i} \succsim^{\theta\setminus\{i\}} \mu_{-i} \succsim^{\theta\setminus\{i\}} \varnothing \Longrightarrow r_i(X_i|\mu'_{-i}) \supseteq r_i(X_i|\mu_{-i}).$$

In words, the choice function of a side satisfies monotone externalities if every agent on this side rejects more contracts when others have a better market condition.[20] The intuition of when this property is satisfied depends on the context. It may be satisfied in settings when agents pool their resources (see Application 4 in Appendix F). For example, when couples share income, a married person may be more selective in accepting an offer as their partner gets a higher-paying job (see Section 3). Monotone externalities may also be satisfied because of competition. A consulting firm may be more likely to reject a candidate based on the prestige of their alma mater when the competing firms have consultants who are graduates of more prestigious schools (see Application 2 in Appendix F).

While monotone externalities is a novel property, it is importantly always satisfied when there are no externalities for side θ because, in that case, the rejection function does not depend on the reference set. Thus, the setting with externalities that we study contains the standard substitutable setting when there are no externalities as a special case.

The conjunction of standard substitutability and monotone externalities is equivalent to the following property.

Definition 4. Choice function C^θ satisfies **substitutability** if for all $i \in \theta$, $X_i, X'_i \subseteq \mathcal{X}_i$, and $\mu_{-i}, \mu'_{-i} \subseteq \mathcal{X}_{-i}$,

$$X'_i \supseteq X_i \text{ and } \mu'_{-i} \succsim^{\theta\setminus\{i\}} \mu_{-i} \succsim^{\theta\setminus\{i\}} \varnothing \Longrightarrow r_i\left(X'_i|\mu'_{-i}\right) \supseteq r_i\left(X_i|\mu_{-i}\right).$$

We refer to this joint condition simply as substitutability because of the parallelism of the monotonicity ideas captured by its two components: standard substitutability captures monotonicity of rejection function with respect to an agent's own choice set, while monotone externalities proxies for such monotonicity with respect to other agents' choice sets. While weaker than the conjunction of standard substitutability and no externalities, our substitutability assumption excludes complementarities. In Section 6, we address the question of which choice

[20]We extend the definitions of consistency and monotone externalities to any $C^{\theta'}$ where $\theta' \subseteq \theta$ by restricting the set of contracts to those associated only with agents in θ'. For any $\theta' \subseteq \theta$, if C^θ satisfies monotone externalities so does $C^{\theta'}$. In addition, if θ' has only one agent, say i, then $C^{\theta'}$ satisfies monotone externalities even if C^θ does not. The reason is our assumption that an agent i's choice conditional on a reference matching μ is the same as the choice conditional on μ_{-i}. This is the only place in the paper that depends on this assumption (cf. Footnote 15).

functions are allowed by providing a characterization of substitutable choice functions in terms of maximizing a set of complete preference orderings.

Let us finish this section with a remark on minimality of a consistent preorder defined as follows: a preorder $\gtrsim^\theta$ is **minimal** if for every consistent preorder $\tilde{\gtrsim}^\theta$, for any $\mu, \mu' \subseteq \mathcal{X}$, $\mu \gtrsim^\theta \mu' \implies \mu \tilde{\gtrsim}^\theta \mu'$. We establish the existence and uniqueness of the minimal preorder in Lemma 4 in Appendix D.[21] Note that whenever substitutability (or monotone externalities) is satisfied for a consistent preorder, then it is also satisfied for the minimal consistent preorder $\gtrsim^\theta$. The reason is that the minimal preorder $\gtrsim^\theta$ compares fewer pairs of reference sets, so substitutability (or monotone externalities) is weaker for the minimal preorder compared to any other consistent preorder.

3 An Application: Couples in a Local Labor Market

In this section, we discuss couples' (or households') labor provision in a local market.[22] Workers play the role of, say, sellers of their labor, and sign contracts with employers, who play the role of buyers. Workers are either single or members of exogenously married couples. As we focus on externalities within couples, we assume that there are no externalities for single workers.

Each worker prefers a higher paying job to a lower paying job. Furthermore, each worker has a *reservation wage*, which is the lowest wage at which a worker is indifferent between accepting a job at this wage and staying unemployed. For single workers, reservation wages are fixed and do not depend on market conditions. However, for married workers reservation wages depend on the incomes of their partner as follows. Within each couple we distinguish between a primary earner and a secondary earner: the labor market participation of the secondary earner depends on the wage of the primary earner.[23] When the primary earner receives a higher wage, the secondary earner becomes more selective. More precisely, the reservation wage of the secondary earner goes up when the primary earner has a higher income. There are no

[21] Because in every preorder $\varnothing \gtrsim^\theta \varnothing$, the minimal preorder is non-empty. Furthermore, consistency implies that even the minimal preorders relate some pairs of distinct matchings provided at least one agent i has at least one contract $x \in \mathcal{X}_i$ such that $c_i(\{x\} | \varnothing) = \{x\}$.

[22] We are grateful to Michael Ostrovsky for suggesting this application. Additional motivating applications—including relative rankings, dynamic matching, profit sharing, and add-ons—are provided in Appendix F. More abstract illustrative examples are provided in Section 4.1.

[23] In this section, we maintain the assumption that the roles of primary earners and secondary earners are fixed and do not depend on market conditions. This assumption is empirically motivated; see the empirical labor market discussion below. We relax this assumption in Appendix E.

externalities for primary earners, so their reservation wages are fixed and do not depend on the income of their partners.

This kind of externality arises in labor markets where members of a couple pool their incomes. For instance, suppose that any secondary earner's job imposes labor-provision disutility c and that the secondary earner is willing to accept the job if any only if it pays wage w such that $U(w+w_p)-c \geq U(w_p)$, where w_p is the wage of the primary earner and U is the concave utility function of income for the couple.[24] In these examples only the wage earned by the primary earner impacts the choice behavior of the secondary earner and the relative locations of the two jobs can be ignored; this is in line with our restriction to local labor markets.

To check substitutability, we define the preorder $\gtrsim_i$ for primary earner i of a couple so that $\mu_i' \gtrsim_i \mu_i$ when the wage specified in contract μ_i' is weakly higher than the wage specified in contract μ_i. For any other worker i, let $\gtrsim_i$ be the trivial preorder for which every pair of contracts is comparable.[25] The better market preorder for workers is consistent with the choice behavior because primary earners choose the contract with the highest wage from any set of contracts; the choice functions satisfy standard substitutability because workers have unit demand; their choice functions satisfy monotone externalities (and hence substitutability) because a secondary earner becomes weakly more selective whenever their partner gets a higher-paying job.

Supposing that employers' choice functions also satisfy substitutability—e.g. because their choice behavior does not exhibit externalities and satisfies standard substitutability—the general theory we develop in subsequent sections implies that a stable job matching exists and is Pareto efficient. The theory also implies that all employers prefer the stable job matching before some set of workers marry to a stable matching following the marriages, while all primary earners prefer a job matching post marriages to the one before; an analogous comparative statics is also valid for divorces.[26]

[24]The utility of income may represent the outcome of intra-household bargaining, as in, e.g., Manser and Brown (1980). The main driver of labor provision costs is hours worked, and the assumption that c is fixed means that different jobs considered by the secondary earner are equivalent in terms of hours worked. Thus the above example is a good approximation of labor markets in which the vast majority of jobs are full-time, as is true, e.g., in Eastern Europe and Russia. For instance, in Bulgaria, the country-wide proportion of full-time jobs was 98.4% in 2019, the most recent year with available OECD data. At the other extreme is, e.g., Switzerland, with only 73.1% of full time jobs. Other than Russia, large economies are in between these two extremes, e.g., the proportion of full-time jobs in the US was 87.6%. The data is available at https://data.oecd.org/emp/part-time-employment-rate.htm.

[25]It is easy to see that these binary relations are preorders.

[26]For existence, see Theorem 1 in Section 4; for efficiency, see Theorem 3 in Section 5; for comparative statics, see Theorem 8 in Appendix G. Note that we can analyze two sides of a market separately because we impose no assumptions relating the choice behavior of agents across sides.

The presence of income-driven externalities within couples has been studied since Becker (1973) and is well documented. The rich literature on the so-called *added worker effect* (e.g. Lundberg (1985), Chiappori (1992), and Cullen and Gruber (2000)) finds that married women are more likely to take or search for paid employment when their husbands are unemployed. Studies based on more recent data—e.g. Kleven, Kreiner and Saez (2009)—relax the distinction between men and women and, instead, like us, analyze couples composed of a primary earner who always participates in the labor market and a secondary earner who chooses whether to work or not.[27]

Finally, note that our restriction to local labor markets plays an important role in the above analysis by decoupling couple's or household's labor provision choices from their decision where to live. This assumption is generally satisfied in labor markets in which members of the working class (also called the middle class) and the poor participate: Their costs of moving or accepting distant jobs are high relative to potential benefits as have been well documented in the empirical studies, see, e.g., Manning and Petrongolo (2017) for a discussion of the UK labor markets and Williams (2017) for an analysis of the US working class. As recognized in this literature, an exception to the ubiquitous locality of labor markets are markets for professional and some managerial jobs—a small fraction of jobs in the economy—which are not necessarily local. The externalities faced by the participants of non-local labor markets, are more complex than those studied in our model and the empirical literature on secondary earners' labor provision discussed above. For instance, the primary earner's choice between jobs in the UK and US, or between jobs on the East Coast and West Coast of the US, would affect the secondary earner's preferences between jobs in these countries or regions.[28]

4 Stable Matchings

As in classical matching theory, a key step in proving the existence of a stable matching is an algorithm akin to the deferred acceptance algorithm.

Our generalization of the deferred acceptance algorithm has two phases. First, we construct an auxiliary matching μ^* such that $C^{\mathcal{S}}(\mathcal{X}|\mu^*) \precsim^{\mathcal{S}} \mu^*$. Then, we use μ^* to construct a stable matching in a way resembling the classic deferred acceptance algorithm of David Gale and

[27]Other related findings include Johnson and Skinner (1986) who find that women increase their labor supply prior to divorce; an evidence that their labor supply was lowered by high earnings of the spouse, an externality of the type we study.

[28]For an analysis of location choices, see e.g. Costa and Kahn (2000) and Compton and Pollak (2007).

Lloyd S. Shapley (1962) and, particularly, its extension by Hatfield and Milgrom (2005): we run the algorithm in rounds, $t = 1,2,\ldots$. In any round $t \geq 1$, we denote by $A^s(t)$ and $A^b(t)$ the set of contracts that are available to the sellers and buyers, respectively. Therefore, the set of contracts held at the beginning of each round is $A^s(t) \cap A^b(t)$. We also track the reference sets for each side: $\mu^s(t)$ is the seller reference set and $\mu^b(t)$ is the buyer reference set.[29]

Phase 1: Construction of an auxiliary matching μ^* such that $\mu^* \gtrsim^{\mathcal{S}} C^{\mathcal{S}}(\mathcal{X}|\mu^*)$. Set $\mu_0 \equiv \emptyset$ and define recursively $\mu_k \equiv C^{\mathcal{S}}(\mathcal{X}|\mu_{k-1})$ for every $k \geq 1$. Since the number of contracts is finite, so is the number of sets of contracts. Therefore, there exist m and $n \leq m$ such that $\mu_{m+1} = \mu_n$. Let $m^* = \min\{m|\exists n \leq m \text{ s.t. } \mu_{m+1} = \mu_n\}$. Let $\mu^* \equiv \mu_{m^*}$. In the proof of Theorem 1, we establish that $\mu^* \gtrsim^{\mathcal{S}} C^{\mathcal{S}}(\mathcal{X}|\mu^*)$.

Phase 2: Construction of a stable matching. Set $A^s(1) \equiv \mathcal{X}$ (all contracts are available to the sellers), $A^b(1) \equiv \emptyset$ (no contracts are available to the buyers), and the reference sets are $\mu^s(1) \equiv \mu^*$, and $\mu^b(1) \equiv \emptyset$. In each round $t = 1,2,\ldots$, we update these sets and matchings as follows:

$$\begin{aligned}
A^s(t+1) &\equiv \mathcal{X} \setminus R^{\mathcal{B}}\big(A^b(t)|\mu^b(t)\big),\\
A^b(t+1) &\equiv \mathcal{X} \setminus R^{\mathcal{S}}\big(A^s(t)|\mu^s(t)\big),\\
\mu^s(t+1) &\equiv C^{\mathcal{S}}\big(A^s(t)|\mu^s(t)\big), \text{ and}\\
\mu^b(t+1) &\equiv C^{\mathcal{B}}\big(A^b(t)|\mu^b(t)\big).
\end{aligned}$$

Thus, the buyers reject some of the contracts available in $A^b(t)$ conditional on the reference set $\mu^b(t)$ and the set of contracts not rejected by the buyers is available to the sellers in the next round, i.e., $A^s(t+1) = \mathcal{X} \setminus R^{\mathcal{B}}\big(A^b(t)|\mu^b(t)\big)$. Likewise, the sellers reject some contracts available in $A^s(t)$ conditional on the reference set $\mu^s(t)$ and the set of contracts that are not rejected by the sellers is available to the buyers in the next round, i.e., $A^b(t+1) = \mathcal{X} \setminus R^{\mathcal{S}}\big(A^s(t)|\mu^s(t)\big)$. We also update the reference sets: at the next round, the sellers' refer-

[29] The tracking of reference sets has no counterpart in earlier formulations of the deferred acceptance algorithms of, among many others, David Gale and Lloyd S. Shapley (1962), Roth (1984), Adachi (2000), Fleiner (2003), Echenique and Oviedo (2004), Hatfield and Milgrom (2005), Echenique and Oviedo (2006), Echenique and Yenmez (2007), Ostrovsky (2008), Hatfield and Kojima (2010), and Bando (2014). In these papers, there is no need to track reference sets and the deferred acceptance algorithm terminates when there are no more rejections and no new offers. However, in our setting, the lack of rejections and new offers is not sufficient to stop the algorithm and we need to run it until the reference sets converge. We run the algorithm in a symmetric way: in each round agents on both sides respond to the offers and rejections from the previous round. This is formally different from the standard approach where agents on the proposing side respond to rejections from the earlier round but the agents on the accepting side respond to offers in the current round. This difference is not substantive: we could run the deferred acceptance algorithm in the latter manner with straightforward adjustments.

ence set is the set of contracts that sellers choose from $A^s(t)$ conditional on $\mu^s(t)$ and likewise for the buyers. We continue updating these sets until round T such that $A^s(T+1) = A^s(T)$, $A^b(T+1) = A^b(T)$, $\mu^s(T+1) = \mu^s(T)$, and $\mu^b(T+1) = \mu^b(T)$. The outcome of the algorithm is then $A^s(T) \cap A^b(T)$.

This is the seller-proposing version of the deferred-acceptance algorithm. The buyer-proposing version can be defined analogously. The main result of this section establishes that the algorithm terminates at some round despite the presence of externalities and, furthermore, it produces a stable matching.

Theorem 1. *(**Sufficiency**) Suppose that the choice functions satisfy substitutability. Then, the algorithm terminates at some finite round T, its outcome $A^s(T) \cap A^b(T)$ is stable, and*

$$\mu^s(T) = \mu^b(T) = A^s(T) \cap A^b(T).$$

The proof relies on monotonicity properties of deferred acceptance transformation—and in that it resembles other such proofs in the matching literature—but we need to address two complications that are specific to the setting with externalities. First, the second phase of our deferred acceptance procedure is monotonic only in some circumstances; it is the role of the first phase to guarantee monotonicity of the second phase. Second, while the no-externalities literature relies on Tarski's fixed-point theorem (e.g., Adachi, 2000), we cannot do so because we work with preorders rather than partial orders and the domain of the function that we analyze is not a lattice. Instead, we use finiteness of the set of contracts to show that the iterative application in the second phase must have two rounds at which the reference matchings are equivalent in the preorder, $\mu^s \sim^{\mathcal{S}} \tilde{\mu}^s$ and $\mu^b \sim^{\mathcal{B}} \tilde{\mu}^b$, while the set of contracts available to the buyers and sellers are the same, $A^s = \tilde{A}^s$ and $A^b = \tilde{A}^b$. Substitutability then implies that $C^{\mathcal{S}}(A^s|\mu^s) = C^{\mathcal{S}}(\tilde{A}^s|\tilde{\mu}^s)$ and $C^{\mathcal{B}}(A^b|\mu^b) = C^{\mathcal{B}}(\tilde{A}^b|\tilde{\mu}^b)$, thereby both the set of available contracts and the reference sets have to be identical in the subsequent rounds implying that the second phase converges. Once the deferred-acceptance algorithm converges, it produces a stable matching. The proof of the last claim relies on a fixed-point characterization of stable matchings presented in Appendix C whereas the details of the proof are provided in Appendix D.

Next we provide a result which shows that monotone externalities is necessary for the existence of a stable matching in a "maximal domain" sense when standard substitutability is satisfied. In this result, we restrict attention to the minimal preorder.

Theorem 2. *(**Necessity**) Suppose that there exists an agent i on side θ such that c_i has externalities and satisfies standard substitutability. Then, there exist substitutable choice functions for the other agents on side θ and substitutable choice functions without externalities for agents on side $-\theta$ such that no stable matching exists.*

Notice that in this theorem the choice function c_i is fixed while choice functions of other agents are constructed. In the construction, $C^{\theta\setminus\{i\}}$ and $C^{-\theta}$ satisfy substitutability but C^{θ} does not.

To develop the intuition for the proof, consider a simple example with two workers i and j on side θ and one firm k on side $-\theta$. For each worker-firm pair there is only one contract; in particular, each worker's choice satisfies standard substitutability. The firm wants to hire as many workers as possible; the firm's choice thus exhibits no externalities and satisfies substitutability. Worker i's choice function exhibits externalities and thus whether worker i wants to work or not depends on whether worker j is hired by the firm or not. These externalities might take one of two forms.

One possibility is that worker i wants to work for the firm only when worker j also works for it. Let then worker j be willing to work only when worker i is not working; this choice of worker j is substitutable and, with the set of workers other than i having only one member, it satisfies monotone externalities (cf. Footnote 20). There is, however, no stable matching because worker j blocks the matching in which both workers are employed, worker i (or worker i and the firm) blocks the matching in which exactly one worker is employed, and worker j and the firm block the matching in which no workers are employed.

The other possibility is that worker i wants to work for the firm only when worker j does not work for the firm. In this case, let worker j be willing to work only when worker i is working. The analysis of this case is analogous to the previous one: our assumptions are satisfied on the submarket without worker i but there does not exist a stable matching.

When there are no externalities, Hatfield and Kominers (2017) show that standard substitutability is a necessary condition for the existence of a stable matching in many-to-many matching markets. In contrast, we assume standard substitutability and show that monotone externalities is a necessary condition for the existence of a stable matching in many-to-many matching markets when there are externalities.

4.1 Illustrative Examples

In this section, we provide two examples to illustrate the deferred acceptance algorithm. In Example 1, substitutability is satisfied, so the algorithm produces a stable matching. In Example 2, substitutability is not satisfied and a stable matching does not exist.

Like the standard deferred acceptance algorithm, in each round of phase 2, substitutability implies that $A^s(t+1) \subseteq A^s(t)$ and $A^b(t+1) \supseteq A^b(t)$, i.e., the sellers make more offers to the buyers while the buyers reject more contracts with each passing round (Lemma 2). As a consequence, the sellers' reference set gets worse and the buyers' reference set gets better. Hence, both of these two sets converge at some round t; however, the algorithm does not necessarily terminate when $A^s(t+1) = A^s(t)$ and $A^b(t+1) = A^b(t)$. Indeed, because of externalities, the set of contracts held at such a round, $A^s(t) \cap A^b(t)$, is not necessarily stable. Instead, the algorithm converges only when $A^s(t+1) = A^s(t)$, $A^b(t+1) = A^b(t)$, $\mu^s(t+1) = \mu^s(t)$, and $\mu^b(t+1) = \mu^b(t)$. The set of contracts held at such a round, $A^s(t) \cap A^b(t)$, is stable.

The next example illustrates this point and shows the steps of the algorithm. It also demonstrates that our algorithm can be viewed as an ascending auction in the presence of externalities.

Example 1. Suppose that there are two sellers s_1 and s_2 and two buyers b_1 and b_2. Seller s_1 and buyer b_1 can sign contract x_1 and seller s_1 and buyer b_2 can sign contract x_2. Seller s_2 can sign contract x_3 with buyer b_2 only.[30] The contractual structure is demonstrated in Figure 1.

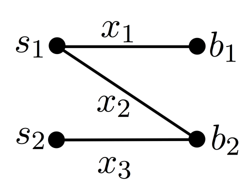

Figure 1: Contractual structure in Example 1.

Seller choice functions do not have externalities. Seller s_1 always chooses one contract, if there exists one, and prefers contract x_2 over x_1 and seller s_2 chooses contract x_3 when it is available. Therefore, seller choice functions satisfy standard substitutability. They also satisfy monotone externalities because there are no externalities for sellers.

Buyer b_1 chooses contract x_1 regardless of the contracts signed by buyer b_2. Conditional on the empty set, buyer b_2 chooses one contract only and prefers contract x_3 to x_2. Conditional

[30]This example is a special case of Application 1 with the following interpretation. Sellers are firms and buyers are workers. Buyers b_1 and b_2 are married. Buyer b_1 is a woman; her choice function does not have externalities. Buyer b_2 is a man and the outside option of not working is ranked higher whenever his wife works. In particular, contract x_2 is ranked below the outside option if the wife has a job.

on the reference set $\{x_1\}$, buyer b_2 chooses contract x_3, if it is available, and rejects x_2, if it is available. Therefore, the only choice function that has externalities is that of buyer b_2, which is summarized by the following table.

	$\{x_2,x_3\}$	$\{x_3\}$	$\{x_2\}$	$\varnothing$
$c_{b_2}(\cdot\|\{x_1\})$	$\{x_3\}$	$\{x_3\}$	$\varnothing$	$\varnothing$
$c_{b_2}(\cdot\|\varnothing)$	$\{x_3\}$	$\{x_3\}$	$\{x_2\}$	$\varnothing$

Table 1: Choice function of buyer b_2 in Example 1. Columns are indexed by the set of available contracts and rows are indexed by the reference set of contracts signed by buyer b_1.

First let us construct the better market condition for buyers. Since buyer b_1 chooses contract x_1 whenever it is available, we have $\{x_1\} \gtrsim_{b_1} \varnothing$. For buyer b_2, using consistency on sets of contracts $\{x_2,x_3\} \supseteq \{x_2\} \supseteq \varnothing$ with the empty set as a reference set, we get $\{x_3\} \gtrsim_{b_2} \{x_2\} \gtrsim_{b_2} \varnothing$. In addition, since $\{x_1\} \gtrsim_{b_1} \varnothing$, $c_{b_2}(\{x_2\}|\{x_1\}) = \varnothing$, and $c_{b_2}(\{x_2\}|\varnothing) = \{x_2\}$, we get $\varnothing \gtrsim_{b_2} \{x_2\}$. Therefore, for buyer b_2, $\{x_3\} \gtrsim_{b_2} \{x_2\} \sim_{b_2} \varnothing$. The better market condition for buyers $\gtrsim^{\mathcal{B}}$ is then defined as $\mu' \gtrsim^{\mathcal{B}} \mu \Leftrightarrow \mu'_{b_i} \gtrsim_{b_i} \mu_{b_i}$ for every $i \in \{1,2\}$. For example, $\{x_1,x_2\} \gtrsim^{\mathcal{B}} \{x_1\}$ because $\{x_1\} \gtrsim_{b_1} \{x_1\}$ and $\{x_2\} \gtrsim_{b_2} \varnothing$. Similarly, $\{x_1\} \gtrsim^{\mathcal{B}} \{x_2\}$ because $\{x_1\} \gtrsim_{b_1} \varnothing$ and $\varnothing \gtrsim_{b_2} \{x_2\}$.

It is easy to check that standard substitutability is satisfied for the buyers. To check monotone externalities, note that choice function of buyer b_1 does not have externalities, so it does not depend on the reference set and the choice function of buyer b_2 rejects more contracts when it is conditional on the reference set $\{x_1\}$ rather than the reference set $\varnothing$, where $\{x_1\} \gtrsim_{b_1} \varnothing$.

Since the choice functions satisfy substitutability, the deferred-acceptance algorithm produces a stable matching (Theorem 1). We now show how it works in this example. In the first phase, we start with $\mu_0 = \varnothing$. Then, $\mu_1 = C^{\mathcal{S}}(\mathcal{X}|\mu_0) = \{x_2,x_3\}$, and $\mu_2 = C^{\mathcal{S}}(\mathcal{X}|\mu_1) = \{x_2,x_3\}$. Since $\mu_1 = \mu_2$, we set $\mu^* = \{x_2,x_3\}$.

In the first round of the second phase, all contracts are available to the sellers, so they choose $\{x_2,x_3\}$. However, no contract is available to the buyers, so they choose the empty set. Therefore, in the second round, the seller reference set is $\{x_2,x_3\}$ and the buyer reference set is the empty set. In addition, the set of contracts available to the buyers is the set of contracts not rejected by the sellers at the first round, which is $\{x_2,x_3\}$.

The algorithm continues to proceed in this way. Table 2 shows all the rounds. Notice that between the fourth and fifth rounds the sets of contracts available to the buyers and sellers are the same, i.e., $A^b(4) = A^b(5)$ and $A^s(4) = A^s(5)$. In the standard deferred acceptance algorithm, we could stop the algorithm here. In our setting, the deferred acceptance does not converge yet because the reference sets for the buyers are different at these two rounds.

	$A^s(t)$	$A^b(t)$	$\mu^s(t)$	$\mu^b(t)$	$C^{\mathcal{S}}(A^s(t)\vert\mu^s(t))$	$C^{\mathcal{B}}(A^b(t)\vert\mu^b(t))$
$t=1$	$\mathcal{X}$	$\varnothing$	$\{x_2,x_3\}$	$\varnothing$	$\{x_2,x_3\}$	$\varnothing$
$t=2$	$\mathcal{X}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\varnothing$	$\{x_2,x_3\}$	$\{x_3\}$
$t=3$	$\{x_1,x_3\}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\{x_3\}$	$\{x_1,x_3\}$	$\{x_3\}$
$t=4$	$\{x_1,x_3\}$	$\mathcal{X}$	$\{x_1,x_3\}$	$\{x_3\}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$
$t=5$	$\{x_1,x_3\}$	$\mathcal{X}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$
$t=6$	$\{x_1,x_3\}$	$\mathcal{X}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$		

Table 2: Rounds of the Deferred Acceptance Algorithm in Example 1.

The algorithm eventually converges at the sixth round and produces the matching $A^s(6) \cap A^b(6) = \{x_1, x_3\}$, which is stable: It is individually rational for all agents. There is only one potential blocking pair (s_1, b_2) via contract x_2 but they do not block this matching because $x_2 \notin c_{b_2}(\{x_2, x_3\}|\{x_1\})$.

Note that the set of contracts available to the sellers, $A^s(t)$, is shrinking and the set of contracts available to the buyers, $A^b(t)$, is expanding as the algorithm proceeds. Likewise, the seller reference set $\mu^s(t)$ is getting worse for the sellers and the buyer reference set $\mu^b(t)$ is getting better for the buyers. ■

When choice functions satisfy standard substitutability, DA produces a stable matching *if* it converges even if monotone externalities is not satisfied (see Theorem 7 and Lemma 3 in Appendix C). However, when monotone externalities fails, it does not have to converge and a stable matching need not exist. We show these two claims with the following example.

Example 2. We modify Example 1 by changing the choice function of buyer b_2. Buyer b_2 chooses all available contracts conditional on the reference set $\{x_1\}$. Furthermore, conditional on the empty set, she chooses contract x_3, if it is available, and rejects x_2, if it is available. Choice function of buyer b_2 is summarized by the following table.

	$\{x_2,x_3\}$	$\{x_3\}$	$\{x_2\}$	$\varnothing$
$c_{b_2}(\cdot\vert\{x_1\})$	$\{x_2,x_3\}$	$\{x_3\}$	$\{x_2\}$	$\varnothing$
$c_{b_2}(\cdot\vert\varnothing)$	$\{x_3\}$	$\{x_3\}$	$\varnothing$	$\varnothing$

Table 3: Choice function of buyer b_2 in Example 2. Columns are indexed by the set of available contracts and rows are indexed by the reference set of contracts signed by buyer b_1.

As in the previous example, it is easy to check that standard substitutability is satisfied for buyers. However, monotone externalities fails. To see this, note that for any consistent preorder we need $\{x_1\} \succsim_{b_1} \varnothing$. But conditional on $\{x_1\}$, buyer b_2 accepts more contracts than conditional on the empty set when the available set of contracts is $\{x_2, x_3\}$.

While our general result implies that there exists a stable matching in Example 1, it is easy to see that there is no stable matching in Example 2: Matchings $\varnothing$ and $\{x_3\}$ are blocked by seller s_1 and buyer b_1 via contract x_1. Matchings $\{x_1\}$ and $\{x_1,x_2\}$ are blocked by seller s_2 and buyer b_2 via contract x_3. Matchings $\{x_2\}$ and $\{x_2,x_3\}$ are not individually rational for buyer b_2. Matching $\{x_1,x_3\}$ is blocked by seller s_1 and buyer b_2 via contract x_2. The last remaining matching, $\mathcal{X}$, is not individually rational for seller s_1.

Now let us consider the deferred-acceptance algorithm. The first phase works as in the previous example since seller choice functions satisfy substitutability. The algorithm starts diverging after round five of the second phase because conditional on the reference set $\mu^b(5) = \{x_1,x_3\}$, the buyers choose all contracts. Table 4 shows the first nine rounds of DA.

	$A^s(t)$	$A^b(t)$	$\mu^s(t)$	$\mu^b(t)$	$C^{\mathcal{S}}(A^s(t)\|\mu^s(t))$	$C^{\mathcal{B}}(A^b(t)\|\mu^b(t))$
$t=1$	$\mathcal{X}$	$\varnothing$	$\{x_2,x_3\}$	$\varnothing$	$\{x_2,x_3\}$	$\varnothing$
$t=2$	$\mathcal{X}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\varnothing$	$\{x_2,x_3\}$	$\{x_3\}$
$t=3$	$\{x_1,x_3\}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\{x_3\}$	$\{x_1,x_3\}$	$\{x_3\}$
$t=4$	$\{x_1,x_3\}$	$\mathcal{X}$	$\{x_1,x_3\}$	$\{x_3\}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$
$t=5$	$\{x_1,x_3\}$	$\mathcal{X}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$	$\mathcal{X}$
$t=6$	$\mathcal{X}$	$\mathcal{X}$	$\{x_1,x_3\}$	$\mathcal{X}$	$\{x_2,x_3\}$	$\mathcal{X}$
$t=7$	$\mathcal{X}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\mathcal{X}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$
$t=8$	$\mathcal{X}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\{x_3\}$
$t=9$	$\{x_1,x_3\}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\{x_3\}$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		

Table 4: Rounds of the Deferred Acceptance Algorithm in Example 2.

At round nine, we get the same sets of contracts available to the buyers and sellers and the same reference sets as in round three. Therefore, the algorithm does not converge. This outcome is not surprising because we showed that there is no stable matching in this example.

■

5 Properties of Stable Matchings under Externalities

Two key normative properties in the standard theory of stable matchings is Pareto efficiency of stable matchings and the existence of side-optimal stable matchings. In this section, we extend them to settings with externalities.

Pareto efficiency extends to our setting as follows.

Theorem 3. *(**Pareto Efficiency**) Suppose that the choice functions satisfy standard substitutability. If matching μ is stable then it is Pareto efficient in the following sense: there is no other matching $\nu \neq \mu$ such that $\nu_i = c_i(\nu \cup \mu|\mu)$ for every agent i.*

The argument in the proof resembles a similar idea in the no-externalities case. We prove a stronger result in Appendix B (Proposition 1).

The counterpart of the side-optimal stable matchings in the setting with externalities is more subtle and it is given by the following result. Before stating this result, we define the following concepts.

Definition 5. A stable matching μ is θ-**optimal** if $\mu \succsim^\theta \mu'$ for every stable matching μ', it is θ-**pessimal** if $\mu \precsim^\theta \mu'$ for every stable matching μ'.

In the standard stable matching theory without externalities, side optimality is measured with respect to the Blair order. This standard result is subsumed.

Theorem 4. *(**Side Optimality**) Suppose that the choice functions satisfy substitutability and, in addition, for side θ there exists a matching $\bar{\mu}^\theta$ such that for any $\mu, X \subseteq \mathcal{X}$, we have $\bar{\mu}^\theta \succsim^\theta C^\theta(X|\mu)$. Then, the θ-proposing deferred-acceptance algorithm when the reference set for side θ is $\bar{\mu}^\theta$ produces a θ-optimal stable matching, which is also a $-\theta$-pessimal stable matching.*

The assumption that there exists a matching $\bar{\mu}^\theta$ such that for any matching $\mu, X \subseteq \mathcal{X}$, $\bar{\mu}^\theta \succsim^\theta C^\theta(X|\mu)$ plays a crucial role in the proof of Theorem 4. It is not innocuous but it is satisfied in our applications provided in Section 3 and Appendix F. In the absence of externalities, this assumption is automatically satisfied because $\succsim^\theta$ is the Blair order. Indeed, for this special case, we can take $\bar{\mu}^\theta$ to be $C^\theta(\mathcal{X})$. Then for any $X \subseteq \mathcal{X}$,

$$\mathcal{X} \supseteq \bar{\mu}^\theta \cup C^\theta(X) \supseteq C^\theta(\mathcal{X}) = \bar{\mu}^\theta$$

and the irrelevance of rejected contracts yield $C^\theta(\bar{\mu}^\theta \cup C^\theta(X)) = C^\theta(\mathcal{X}) = \bar{\mu}^\theta$. This implies $\bar{\mu}^\theta \succsim^\theta C^\theta(X)$ for any X. Thus, Theorem 4 subsumes the standard insight that, in the absence of externalities, θ-proposing deferred acceptance algorithm produces the θ-optimal stable matching with respect to the Blair order if choice functions satisfy substitutability. This matching is also $(-\theta)$-pessimal.

Before we end this section, we provide an example which shows the displayed assumption in Theorem 4 is necessary. In addition, this example also shows that the set of stable matchings need not have a lattice structure even when choice functions satisfy substitutability.[31]

Example 3. Suppose that there are two buyers b_1, b_2 and one seller, s_1. There is only one contract associated with every seller-buyer pair. Let the contract between b_1 and s_1 be x_1 and the contract between b_2 and s_1 be x_2. Since there is only one seller, there are no externalities for the seller side.

Choice functions are as follows: Seller s_1 chooses all contracts available. Buyer b_1 chooses x_1 conditional on the reference set $\{x_2\}$ and rejects x_1 conditional on the empty set. Buyer b_2 chooses x_2 conditional on the reference set $\{x_1\}$ and rejects x_2 conditional on the empty set. That is each buyer chooses their contract only if the other buyer has the other contract.

Choice function of the seller satisfies substitutability. For buyers, consider the preorder $\succsim^{\mathcal{B}}$ with the domain $\{\varnothing\}$ such that $\varnothing \succsim^{\mathcal{B}} \varnothing$. Thus this preorder does not compare any other pairs of matchings.[32] This preorder is consistent because conditional on the empty set both buyers do not choose any contract. In addition, the buyer-side choice function satisfies substitutability because the buyer-side rejection function is monotone conditional on the empty set.

There exists no buyer-optimal stable matching in this example because both the empty set and $\{x_1, x_2\}$ are stable matchings, which cannot be compared by the preorder $\succsim^{\mathcal{B}}$. This is compatible with Theorem 4 because there exists no buyer-optimal matching $\bar{\mu}^{\mathcal{B}}$ such that $\bar{\mu}^{\mathcal{B}} \succsim^{\mathcal{B}} C^{\mathcal{B}}(X|\mu)$ for all matchings μ, X, which is the additional assumption needed for the existence of a buyer-optimal stable matching. In addition, the set of stable matchings does not have a lattice structure. ■

Remark 1. Suppose that agents are members of coalitions and coordinate their choices. Examples include couples, sports teams, corporate divisions, single firms, or even multiple firms controlled by the same owner. If an outside observer is unaware that the coalition—rather than the agents—is the decision maker, the outside observer might infer that the choices of the coalition members exhibit externalities. The standard matching theory without externalities guarantees the existence of stable matchings and their properties among such coalitions provided coalitional choice functions satisfy the standard substitutes condition (Hatfield and Milgrom, 2005; Hatfield and Kominers, 2017). In particular, the standard theory guarantees the existence of stable matchings that are side-optimal for the coalitions. As the above example

[31] In contrast, when there are no externalities, standard substitutability implies that the set of stable matchings is a lattice (Hatfield and Milgrom, 2005). Such a structure may also exist in our setting under additional assumptions. We leave this question for future research.

[32] In general, we allow the domain of the preorder to be smaller than the set of all matchings, which is the case in this example.

shows, in our framework, the existence of side-optimal stable matchings is not guaranteed, and indeed, the above example cannot be reinterpreted as coalitional choice where buyers form a coalition with a choice function $C^{\mathcal{B}}$ that has no externalities: To have $\{x_1, x_2\}$ as a stable matching as in the example, we need the coalitional choice to satisfy $C^{\mathcal{B}}(\{x_1, x_2\}) = \{x_1, x_2\}$. Then substitutability implies that $C^{\mathcal{B}}(X) = \{X\}$ for every $X \subseteq \{x_1, x_2\}$. Therefore, every matching is stable with this coalitional choice unlike the example above which has only two stable matchings.

6 A Characterization of Substitutable Choice Functions

Which choice functions are substitutable? We establish a simple structure of substitutable choice functions. We describe the structure using the standard matching concept of truncation (see Roth and Rothblum (1999)). Linear order $\succ'$ over $\mathcal{X}_i \cup \{\emptyset\}$ is a **truncation** of linear order $\succ$ over $\mathcal{X}_i \cup \{\emptyset\}$ if, for all $x, y \in \mathcal{X}_i$ the following two implications hold true:

- $x \succ' \emptyset$ implies $x \succ \emptyset$, and
- $x \succ' y \succ' \emptyset$ implies $x \succ y \succ \emptyset$.

In words, any contract ranked above the empty set by the linear order $\succ'$ is also ranked above the empty set by the linear order $\succ$ and the relative ranking of any two contracts preferred to the empty set in the linear order $\succ'$ is the same as in the linear order $\succ$. Therefore, a truncation of a linear order moves the outside option $\emptyset$ higher in the ranking.

The next result characterizes choice functions satisfying our substitutability condition.

Theorem 5. *(**Characterization of Substitutability**) Choice function C^{θ} satisfies substitutability if, and only if, for every agent $i \in \theta$ there is a nonempty set $\mathcal{J}$ and linear orders $\succ_j^{\mu_{-i}}$ over $\mathcal{X}_i \cup \{\emptyset\}$ indexed by $j \in \mathcal{J}$ and matching μ_{-i} that does not include i's contracts such that if $\mu'_{-i} \succsim^{\theta} \mu_{-i} \succsim^{\theta} \emptyset$ then for any $j \in \mathcal{J}$, $\succ_j^{\mu'_{-i}}$ is a truncation of $\succ_j^{\mu_{-i}}$. Furthermore, for any $X, \mu \subseteq \mathcal{X}$,*

$$c_i(X_i|\mu_{-i}) = \bigcup_{j \in \mathcal{J}} \{x_j^{\mu_{-i}}\},$$

where $x_j^{\mu_{-i}}$ is the maximum element of $X_i \cup \{\emptyset\}$ in order $\succ_j^{\mu_{-i}}$.

This result is inspired by the Aizerman and Malishevski (1981) decomposition result for substitutable functions when there are no externalities. It states that the choice function can be constructed from a set of linear orders over individual contracts such that the choice from a set conditional on a reference set is the union of the most-preferred contracts with respect to these linear orders. In this representation, the linear orders depend on the reference set and as the reference set gets better with respect to the better market condition the linear orders are truncated.[33]

Theorem 5 takes a particularly simple form in the context of the local labor market model of Section 3. In the simplest version of this model, each couple in the labor market consists of a primary and a secondary earner. The choices of a primary earner exhibits no externalities and hence any choice function of a primary earner satisfies our substitutes condition. Choices of a secondary earner can exhibit externalities and the choice function of a secondary satisfies the substitutes condition if and only if it is represented by a family of rankings indexed by the contract of the primary earner and these rankings only differ in how being unemployed is ranked: the higher the wage of the primary earner is, the higher is the reservation wage of the secondary earner.

7 Conclusion

In this paper, we have studied a two-sided matching problem with externalities where each agent's choice depends on other agents' contracts. For such settings, we have developed the theory of stable matchings by introducing a new substitutability condition when externalities are present. More explicitly, we have studied the existence of stable matchings, Pareto efficiency of stable matchings, side-optimal stable matchings, the deferred acceptance algorithm, and the rural hospitals theorem (which is in Appendix A). Unlike the previous matching literature, we have not relied on fixed point theorems; instead, we have used elementary techniques to overcome the difficulties associated with externalities.

The standard substitutability condition can be weakened without affecting our results in two different ways. In the first approach, the reference set can be restricted to be a set that can be chosen by side θ. More formally, consider the minimal set of matchings $\mathcal{A}^\theta$ that contains the empty set and satisfies $C^\theta(X|\mu) \in \mathcal{A}^\theta$ whenever $X \subseteq \mathcal{X}$ and $\mu \in \mathcal{A}^\theta$. The minimal such

[33] Can we interpret rankings $\succ_j^{\mu_{-i}}$ in this theorem as preferences of sub-agents for agent i? Such an interpretation runs into the problem that two or more of the sub-agents might rank the same contract x as their best contract from a choice set and, in general, it is not possible to designate one of these subagents to be the signatory of x. In fact, Remark 1 above shows that—despite Theorem 5—our conditions cannot be in general reinterpreted as coalitional choices. We would like to thank an anonymous referee for raising the question.

domain is $\mathcal{A}^\theta \equiv \underset{t=0,1,\ldots}{\cup} \mathcal{A}_t^\theta$ where $\mathcal{A}_0^\theta \equiv \{\varnothing\}$ and $\mathcal{A}_t^\theta$ for $t \geq 1$ are defined recursively

$$\mathcal{A}_t^\theta \equiv \{C^\theta(X|\mu) : X \subseteq \mathcal{X}, \mu \in \mathcal{A}_{t-1}^\theta\} \cup \mathcal{A}_{t-1}^\theta.$$

Since there exists a finite number of contracts, $\mathcal{A}^\theta$ is well-defined; it is the set of all matchings that can be reached from the empty set by applying the choice function C^θ. Standard substitutability can be weakened by imposing it only for reference sets in $\mathcal{A}^\theta$.

The second approach to weaken standard substitutability works only when agents on one side of the market have unit demand using the techniques developed in Hatfield and Kojima (2010), Hatfield and Kominers (2016), and Hatfield, Kominers and Westkamp (2017) when there are no externalities. These conditions usually proceed by restricting X' and X under which the standard substitutability condition holds. Such conditions can also be studied in our setting when one side of the market can sign at most one contract. Furthermore, a combination of the two approaches can be used when agents on one side of the market have unit demand.

We believe that our notion of substitutability will be useful to study other important questions in matching markets with externalities. For example, the relations between pairwise stability, group stability, core, and other stability concepts have been an important question in classical matching theory at least since Blair (1988). We analyze the relation between pairwise and group stability in Appendix B, but many related questions remain open. The strategy-proofness of deferred acceptance algorithm (for the proposing side) has been another important question extensively studied since Lester E Dubins and David A Freedman (1981). We think that a deferred acceptance procedure remains strategy-proof in our setting provided we impose the law of aggregate demand à la Hatfield and Milgrom (2005); we leave an exploration of this question for future work. Furthermore, even though we have studied two-sided markets, we think that our techniques are applicable to more general markets such as the supply chain networks of Ostrovsky (2008) where externalities may naturally appear.[34]

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Appendix A: Law of Aggregate Demand and the Rural Hospitals Theorem

In this section, we provide a generalization of the law of aggregate demand (Hatfield and Milgrom, 2005) and size monotonicity (Alkan and Gale, 2003). In markets without externalities, this generalization is due to Fleiner (2003). For each contract $x \in \mathcal{X}$, there is a corresponding weight denoted by $w(x) \in \mathbb{R}$. The generalized law of aggregate demand requires that for agent $i \in \theta$ the total weight of contracts chosen from X conditional on μ is weakly smaller than the total weight of contracts chosen from X' conditional on μ' for any $X' \supseteq X$ and $\mu' \gtrsim^{\theta} \mu$. For a set of contracts $X \subseteq \mathcal{X}$, let $w(X) \equiv \sum_{x \in X} w(x)$. We provide a formal definition as follows.

Definition 6. Choice function c_i satisfies **the law of aggregate demand** if $i \in \theta$ and for any $X \subseteq X'$ and $\mu \lesssim^{\theta} \mu'$ then $w(c_i(X|\mu)) \leq w(c_i(X'|\mu'))$.

Previous definitions in the matching literature are restricted to the settings without externalities, and assume that the weight on all contracts are positive and equal (with the only exception of Fleiner (2003)). Under this assumption, the generalized law of aggregate demand reduces to for any $X \subseteq X'$ and $\mu \subseteq \mathcal{X}$, $|c_i(X|\mu)| \leq |c_i(X'|\mu)|$. In terms of the demand metaphor of Hatfield and Milgrom (2005), all contracts are traded at price one. In contrast, we allow any prices.

We study how the weight of contracts changes for an agent in different stable matchings. We show that the weight remains the same regardless of the stable matching. This extends the rural hospitals theorem of Hatfield and Milgrom (2005) in two directions: We allow different contracts to have different weights and also preferences of an agent can depend on contracts signed by others.

Theorem 6. ***(Rural Hospital Theorem)*** *Suppose that choice functions satisfy substitutability and the law of aggregate demand, and that there exists a matching $\bar{\mu}^{\theta}$ such that for any $\mu, X \subseteq$*

$\mathcal{X}$, we have $\bar{\mu}^\theta \gtrsim^\theta C^\theta(X|\mu)$. Then, for any two stable matchings μ and μ', $w(\mu_i) = w(\mu'_i)$ for every agent i.

Proof. Without loss of generality assume that $\theta = s$. Then, by Theorem 4, there exists a stable matching μ^*, which is seller-optimal and buyer-pessimal simultaneously. We show that for any stable matching μ, $w(\mu_i) = w(\mu^*_i)$. As it is shown in the proof of Theorem 4, f has two fixed points $(A^{*s}, A^{*b}, \mu^*, \mu^*)$ and (A^s, A^b, μ, μ) such that $(A^{*s}, A^{*b}, \mu^*, \mu^*) \sqsupseteq (A^s, A^b, \mu, \mu)$. Therefore, $A^{*s} \supseteq A^s$, $A^{*b} \subseteq A^b$, $\mu^* \gtrsim^{\mathcal{S}} \mu$ and $\mu^* \lesssim^{\mathcal{B}} \mu$. Now by the law of aggregate demand for any $i \in S$, $w(c_i(A^{*s}|\mu^*)) \geq w(c_i(A^s|\mu))$, which is equivalent to $w(\mu^*_i) \geq w(\mu_i)$ since $(A^{*s}, A^{*b}, \mu^*, \mu^*)$ and (A^s, A^b, μ, μ) are fixed points of f. When this is summed over all sellers, we get $w(\mu^*) \geq w(\mu)$. Similarly, for any $i \in B$, $w(c_i(A^{*b}|\mu^*)) \leq w(c_i(A^b|\mu))$, which is equivalent to $w(\mu^*_i) \leq w(\mu_i)$ since $(A^{*s}, A^{*b}, \mu^*, \mu^*)$ and (A^s, A^b, μ, μ) are fixed points of f. When summed over all buyers, this implies $w(\mu^*) \leq w(\mu)$. Therefore, $w(\mu^*) = w(\mu)$, moreover, all of the individual inequalities must hold as equalities implying that for any agent i, $w(\mu^*_i) = w(\mu_i)$. ■

Remark 2. In the special case when all weights are strictly positive, under the assumptions of Theorem 6, an agent's choice from the same set conditional on two ranked matchings needs to be the same. Indeed, let $i \in \theta$ be an agent. Suppose that $X, \mu, \mu' \subseteq \mathcal{X}$ are such that $\mu \lesssim^\theta \mu'$. Then, by substitutability, $c_i(X|\mu) \supseteq c_i(X|\mu')$. But the law of aggregate demand implies that $w(c_i(X|\mu)) \leq w(c_i(X|\mu'))$. Since all weights are strictly positive, we get that $c_i(X|\mu) = c_i(X|\mu')$. This argument does not mean that there are no externalities because the choice conditional on two matchings that are not ranked with respect to $\gtrsim^\theta$ can still be different.

Appendix B: Group Stability

In this section, we provide a definition of a blocking set of contracts and the corresponding definition of *group stability*. Then we show a result relating stable matchings and group stable matchings.

A set $X \subseteq \mathcal{X}$ **blocks** matching μ if $X \nsubseteq \mu$ and for all $i \in \mathcal{I}$ we have $X_i \subseteq c_i(\mu \cup X|\mu)$. Less formally, conditional on matching μ, every agent who is associated with a contract in X wants to sign all contracts in X associated with them. In this case, X is also called a **blocking set** for μ. A matching is **group stable** if it is individually rational for all agents and there is no blocking set of contracts. Without externalities, this stability concept has been used before (see, e.g., Roth, 1984 and Hatfield and Kominers, 2017).

Proposition 1. ***[Equivalence of Stability and Group Stability]*** *Suppose that choice functions satisfy substitutability. Then a matching is stable if, and only if, it is group stable.*

See Roth and Sotomayor (1990); Echenique and Oviedo (2006); Hatfield and Kominers (2017) for earlier developments of this equivalence when there are no externalities. In particular, Hatfield and Kominers (2017) prove the same result when there are no externalities. The same proof works in our setting as well. More precisely, the following lemma is enough to prove the proposition, which only requires standard substitutability.

Lemma 1. *Suppose X blocks matching μ and choice functions satisfy standard substitutability. Then for every $x \in X \setminus \mu$, $\{x\}$ blocks μ.*

Proof. If X is a blocking set, then $X \subseteq C^{\mathcal{S}}(\mu \cup X|\mu) \cap C^{\mathcal{B}}(\mu \cup X|\mu)$. Take any $x \in X \setminus \mu$. Since choice function c_i satisfies standard substitutability, we have $r_i(\mu \cup \{x\}|\mu) \subseteq r_i(\mu \cup X|\mu)$ for every agent i. This implies $x \in c_i(\mu \cup \{x\}|\mu)$ for every i, so $x \in C^{\mathcal{S}}(\mu \cup \{x\}|\mu) \cap C^{\mathcal{B}}(\mu \cup \{x\}|\mu)$. Therefore, $\{x\}$ is a blocking set for μ. ■

Appendix C: Fixed-Point Approach to Stability

Our analysis of the existence of stable matchings builds on the fixed-point methods used in Adachi (2000), Fleiner (2003), Echenique and Oviedo (2004, 2006), Hatfield and Milgrom (2005), Bando (2014), and others. In this section, we construct a function that mimics the iterative step of the deferred-acceptance algorithm and study properties of its fixed points.

Each iteration in the second phase of our deferred-acceptance algorithm can be described as the following transformation function

$$f\left(A^s, A^b, \mu^s, \mu^b\right) \equiv \left(\mathcal{X} \setminus R^{\mathcal{B}}(A^b|\mu^b), \mathcal{X} \setminus R^{\mathcal{S}}(A^s|\mu^s), C^{\mathcal{S}}\left(A^s|\mu^s\right), C^{\mathcal{B}}(A^b|\mu^b)\right),$$

where f is a function from $2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}}$ into itself.

Function f has two important properties, monotonicity and stability of its fixed points, that are captured in the following auxiliary results.

Lemma 2. *Suppose that the choice functions satisfy substitutability. Then function f is monotone increasing with respect to the preorder $\sqsubseteq$ defined as follows:*

$$(A^s, A^b, \mu^s, \mu^b) \sqsubseteq (\tilde{A}^s, \tilde{A}^b, \tilde{\mu}^s, \tilde{\mu}^b) \Longleftrightarrow A^s \subseteq \tilde{A}^s, A^b \supseteq \tilde{A}^b, \mu^s \precsim^{\mathcal{S}} \tilde{\mu}^s, \mu^b \succsim^{\mathcal{B}} \tilde{\mu}^b.$$

Proof. Function f is monotonic in $\sqsubseteq$ because for any $A^s \subseteq \tilde{A}^s, A^b \supseteq \tilde{A}^b, \mu^s \precsim^{\mathcal{S}} \tilde{\mu}^s, \mu^b \succsim^{\mathcal{B}} \tilde{\mu}^b$, substitutability implies that

$$\begin{aligned} \mathcal{X}\backslash R^{\mathcal{B}}(A^b|\mu^b) &\subseteq \mathcal{X}\backslash R^{\mathcal{B}}(\tilde{A}^b|\tilde{\mu}^b), \\ \mathcal{X}\backslash R^{\mathcal{S}}(A^s|\mu^s) &\supseteq \mathcal{X}\backslash R^{\mathcal{S}}(\tilde{A}^s|\tilde{\mu}^s), \end{aligned}$$

and consistency implies that

$$\begin{aligned} C^{\mathcal{S}}(A^s|\mu^s) &\precsim^{\mathcal{S}} C^{\mathcal{S}}(\tilde{A}^s|\tilde{\mu}^s), \\ C^{\mathcal{B}}(A^b|\mu^b) &\succsim^{\mathcal{B}} C^{\mathcal{B}}(\tilde{A}^b|\tilde{\mu}^b). \end{aligned}$$

Therefore, $(A^s, A^b, \mu^s, \mu^b) \sqsubseteq (\tilde{A}^s, \tilde{A}^b, \tilde{\mu}^s, \tilde{\mu}^b)$ implies that $f(A^s, A^b, \mu^s, \mu^b) \sqsubseteq f(\tilde{A}^s, \tilde{A}^b, \tilde{\mu}^s, \tilde{\mu}^b)$. ■

The fixed points of f satisfy the following properties even when the choice functions do not satisfy substitutability or monotone externalities.

Lemma 3. *Let $\left(A^s, A^b, \mu^s, \mu^b\right)$ be a fixed point of function f. Then $A^s \cup A^b = \mathcal{X}$ and*

$$\mu^s = \mu^b = A^s \cap A^b = C^{\mathcal{B}}(A^b|\mu^b) = C^{\mathcal{S}}(A^s|\mu^s).$$

Proof. $A^s \cup A^b = A^s \cup \left[\mathcal{X}\backslash R^{\mathcal{S}}(A^s|\mu^s))\right] \supseteq A^s \cup [\mathcal{X} \setminus A^s] = \mathcal{X}$, so

$$A^s \cup A^b = \mathcal{X}.$$

Similarly, $A^s \cap A^b = A^s \cap \left[\mathcal{X}\backslash R^{\mathcal{S}}(A^s|\mu^s))\right] = A^s \setminus R^{\mathcal{S}}(A^s|\mu^s) = C^{\mathcal{S}}\left(A^s|\mu^s\right)$, which implies $C^{\mathcal{S}}\left(A^s|\mu^s\right) = A^s \cap A^b$. Analogously for buyers, $C^{\mathcal{B}}\left(A^b|\mu^b\right) = A^s \cap A^b$. Finally, $\mu^s = C^{\mathcal{S}}\left(A^s|\mu^s\right)$ and $\mu^b = C^{\mathcal{B}}\left(A^b|\mu^b\right)$ imply

$$\mu^s = \mu^b = A^s \cap A^b = C^{\mathcal{B}}(A^b|\mu^b) = C^{\mathcal{S}}\left(A^s|\mu^s\right).$$

■

When choice functions satisfy standard substitutability, a matching is stable if, and only if, it can be supported as a fixed point of f.

Theorem 7. *(**Characterization of Stability**) Suppose that the choice functions satisfy standard substitutability. Then a matching μ is stable if, and only if, there exist sets of contracts $A^s, A^b \subseteq \mathcal{X}$ such that $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of function f.*

Proof. First, suppose that $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f. Claim 1 below shows that μ is a stable matching.

Claim 1. Suppose that the choice functions satisfy standard substitutability. Then matching μ is stable.

Proof. Suppose, for contradiction, that μ is not stable. Then there are three possibilities, all of which we proceed to rule out.

1. Matching μ is not individually rational for some seller j, that is $c_j(\mu|\mu) \subsetneq \mu_j$. Since $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f, $C^{\mathcal{S}}(A^s|\mu) = \mu$ and $A^s \supseteq \mu$. But standard substitutability and $c_j(\mu|\mu) \subsetneq \mu_j$ imply that there is a contract $x \in \mu_j$ rejected out of A^s by agent j, that is $x \notin C^{\mathcal{S}}(A^s|\mu)$, a contradiction.

2. Matching μ is not individually rational for some buyer i, that is $c_i(\mu|\mu) \subsetneq \mu_i$. This is analogous to the previous case since f treats buyers and sellers symmetrically.

3. There exists a blocking pair $i \in \mathcal{B}$ and $j \in \mathcal{S}$ with contract $x \in \mathcal{X}_i \cap \mathcal{X}_j$ such that $x \notin \mu$ and $x \in c_i(\mu \cup \{x\}|\mu) \cap c_j(\mu \cup \{x\}|\mu)$. Since $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f, by Lemma 3, $A^s \cup A^b = \mathcal{X}$. Therefore, without loss of generality, assume that $x \in A^b$. Again, since $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f, by Lemma 3, $C^{\mathcal{B}}(A^b|\mu) = \mu$, which implies that $c_i(A^b|\mu) = \mu_i$. By the irrelevance of rejected contracts, for any set Y such that $A^b \supseteq Y \supseteq \mu$, $c_i(Y|\mu) = \mu_i$. In particular, for $Y = \mu \cup \{x\}$, $c_i(\mu \cup \{x\}|\mu) = \mu_i$, which is a contradiction because $x \in c_i(\mu \cup \{x\}|\mu) \setminus \mu$.

To finish the proof of the theorem, we need to show that if matching μ is stable then there exist sets of contracts A^s, A^b such that $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f. The following is useful in our construction of A^s and A^b.

Claim 2. Suppose that the choice functions satisfy standard substitutability. Then the function $M^\theta(\mu) \equiv \max\{X \subseteq \mathcal{X} | C^\theta(X|\mu) = \mu\}$, where the maximum is with respect to set inclusion, is well defined. Moreover, for any contract $x \notin M^\theta(\mu)$, $x \in C^\theta(M^\theta(\mu) \cup x|\mu)$.

Proof. If there are two sets M' and M'' such that $C^\theta(M'|\mu) = C^\theta(M''|\mu) = \mu$, then (by standard substitutability)

$$
\begin{aligned}
C^{\theta}(M' \cup M''|\mu) &= (M' \cup M'') \setminus R^{\theta}(M' \cup M''|\mu) = \left[M' \setminus R^{\theta}(M' \cup M''|\mu)\right] \cup \left[M'' \setminus R^{\theta}(M' \cup M''|\mu)\right] \\
&\subseteq \left[M' \setminus R^{\theta}(M'|\mu)\right] \cup \left[M'' \setminus R^{\theta}(M''|\mu)\right] = \mu.
\end{aligned}
$$

If $C^{\theta}(M' \cup M''|\mu)$ was a proper subset of μ, then the irrelevance of rejected contracts would imply that $C^{\theta}(M'|\mu) = C^{\theta}(M''|\mu) = C^{\theta}(M' \cup M''|\mu)$, which is a contradiction. Therefore, $M^{\theta}(\mu)$ is well defined. Let $x \notin M^{\theta}(\mu)$. If $x \notin C^{\theta}(M^{\theta}(\mu) \cup x|\mu)$, then $C^{\theta}(M^{\theta}(\mu) \cup x|\mu) = C^{\theta}(M^{\theta}(\mu)|\mu)$ by the irrelevance of rejected contracts. But this implies $C^{\theta}(M^{\theta}(\mu) \cup x|\mu) = \mu$, which contradicts maximality of $M^{\theta}(\mu)$. Hence, $x \in C^{\theta}(M^{\theta}(\mu) \cup x|\mu)$.

Claim 3. Suppose that matching μ is stable and the choice functions satisfy standard substitutability. Then there exist sets of contracts A^s and A^b such that $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f.

Proof. By Claim 2, there exists the largest set $M^{\theta}(\mu) = \max\{X \subseteq \mathcal{X} | C^{\theta}(X|\mu) = \mu\}$. Let $A^s \equiv M^{\mathcal{S}}(\mu)$ and $A^b \equiv \mathcal{X} \setminus R^{\mathcal{S}}(A^s|\mu)$. By construction of $M^{\mathcal{S}}(\mu)$, $\mu = C^{\mathcal{S}}(A^s|\mu)$. Thus, we get $A^s \cap A^b = A^s \cap (\mathcal{X} \setminus R^{\mathcal{S}}(A^s|\mu)) = C^{\mathcal{S}}(A^s|\mu) = \mu$. To finish the proof, we need to show $\mu = C^{\mathcal{B}}(A^b|\mu)$ and $A^s = \mathcal{X} \setminus R^{\mathcal{B}}(A^b|\mu)$.

Note that $A^b = \mathcal{X} \setminus R^{\mathcal{S}}(A^s|\mu) = (\mathcal{X} \setminus A^s) \cup C^{\mathcal{S}}(A^s|\mu) = (\mathcal{X} \setminus A^s) \cup \mu$. Therefore, $A^b \supseteq \mu$. If $Y \equiv C^{\mathcal{B}}(A^b|\mu) \neq \mu$, there are two cases, both of which contradict stability of μ. First, if $Y \subsetneq \mu$, then the irrelevance of rejected contracts implies $C^{\mathcal{B}}(\mu|\mu) = Y$, implying that μ is not individually rational for some buyers, contradicting stability. Second, if $Y \nsubseteq \mu$, then there exists $y \in Y \setminus \mu$, and $y \in C^{\mathcal{B}}(\mu \cup \{y\}|\mu)$ by standard substitutability since $y \in C^{\mathcal{B}}(A^b|\mu)$ and $A^b \supseteq \mu \cup \{y\}$. But we also have that $y \in C^{\mathcal{S}}(A^s \cup \{y\}|\mu)$ by Claim 2. Then the agents associated with $\{y\}$ block μ, contradicting stability. Thus, the only case consistent with stability is $C^{\mathcal{B}}(A^b|\mu) = \mu$.

Finally, we show that $A^s = \mathcal{X} \setminus R^{\mathcal{B}}(A^b|\mu) = \mathcal{X} \setminus R^{\mathcal{B}}(\mathcal{X} \setminus R^{\mathcal{S}}(A^s|\mu)|\mu)$. Since $C^{\mathcal{B}}(A^b|\mu) = \mu$, then $\mathcal{X} \setminus R^{\mathcal{B}}(A^b|\mu) = \mathcal{X} \setminus (A^b \setminus \mu) = \mathcal{X} \setminus (((\mathcal{X} \setminus A^s) \cup \mu) \setminus \mu) = \mathcal{X} \setminus (\mathcal{X} \setminus A^s) = A^s$ and we have the result. ∎

Appendix D: Proofs

In this appendix, we provide the omitted proofs.

Minimal Preorder

In Section 2, we defined minimal preorder and asserted its existence and uniqueness. We prove these claims in the next lemma.

Lemma 4. *There exists a unique minimal preorder that is consistent with the side choice function C^θ.*

Proof. Consider the following preorder $\tilde{\gtrsim}_i$ for agent $i \in \theta$: for every $\mu_i, \mu_i' \subseteq \mathcal{X}_i$, $\mu_i \tilde{\gtrsim}_i \mu_i'$. Let $\tilde{\gtrsim}^\theta$ be the corresponding preorder for side θ. Preorder $\tilde{\gtrsim}^\theta$ is consistent with the choice function C^θ because for every $X' \supseteq X$ and $\mu' \tilde{\gtrsim}^\theta \mu$, we have $C^\theta(X'|\mu') \tilde{\gtrsim}^\theta C^\theta(X|\mu)$. Hence, there exists at least one preorder consistent with C^θ. Now, let us construct a minimal one.

Suppose that $\{\gtrsim_1^\theta, \gtrsim_2^\theta, \dots, \gtrsim_k^\theta\}$ is the set of all preorders for side θ that are consistent with choice function C^θ. Define the following binary relation: $\mu' \gtrsim^\theta \mu$ if, and only if, $\mu' \gtrsim_j^\theta \mu$ for every $j = 1, \dots, k$. The binary relation $\gtrsim^\theta$ is reflexive and transitive, so it is a preorder. Furthermore, $\varnothing \gtrsim^\theta \varnothing$ since the same relation holds for each $\gtrsim_j^\theta$ for every $j = 1, \dots, k$.

Now we show that $\gtrsim^\theta$ is consistent with the side choice function C^θ. Let $X' \supseteq X$ and $\mu' \gtrsim^\theta \mu$. Then, by the construction of $\gtrsim^\theta$, $\mu' \gtrsim_j^\theta \mu$ for every $j = 1, \dots, k$. By consistency of $\gtrsim_j^\theta$, we get $C^\theta(X'|\mu') \gtrsim_j^\theta C^\theta(X|\mu)$ for every $j = 1, \dots, k$. As a result, $C^\theta(X'|\mu') \gtrsim^\theta C^\theta(X|\mu)$ by the construction of $\gtrsim^\theta$. Therefore, $\gtrsim^\theta$ is also consistent with the choice function C^θ. Since the number of preorders is finite, this argument shows that there exists a unique minimal preorder $\gtrsim^\theta$ that is consistent with C^θ. ■

Proof of Theorem 1

First, let us consider the first phase of the algorithm and check that $\mu^* \gtrsim^\mathcal{S} C^\mathcal{S}(\mathcal{X}|\mu^*)$. Since $C^\mathcal{S}(\mathcal{X}|\mu_{k-1}) = \mu_k$, by the irrelevance of rejected contracts, we get $C^\mathcal{S}(\mu_k|\mu_{k-1}) = \mu_k$ for every $k \geq 1$. We show that $\mu_k \gtrsim^\mathcal{S} \mu_{k-1}$ for every $k \geq 1$. The proof is by mathematical induction on k. For the base case when $k = 1$, note that $\mathcal{X} \supseteq \varnothing$ and consistency imply that

$$\mu_1 = C^\mathcal{S}(\mathcal{X}|\varnothing) \gtrsim^\mathcal{S} C^\mathcal{S}(\varnothing|\varnothing) = \varnothing = \mu_0.$$

For the general case, $\mu_k \gtrsim^\mathcal{S} \mu_{k-1}$ and $\mathcal{X} \supseteq \mu_k$ imply that (by consistency)

$$\mu_{k+1} = C^\mathcal{S}(\mathcal{X}|\mu_k) \gtrsim^\mathcal{S} C^\mathcal{S}(\mu_k|\mu_{k-1}) = \mu_k.$$

Therefore, $\{\mu_k\}_{k\geq 1}$ is a monotone sequence with respect to the preorder $\succsim^{\mathcal{S}}$. Since the number of contracts is finite, there exists n and $m \geq n$ such that $\mu_{m+1} = \mu_n$; we take the minimum m satisfying this property and set $\mu^* = \mu_m$. Then,

$$C^{\mathcal{S}}(\mathcal{X}|\mu_m) = \mu_{m+1} = \mu_n \precsim^{\mathcal{S}} \mu_m$$

where the monotonicity comparison follows because $\precsim^{\mathcal{S}}$ is transitive.

It remains to show that the second phase converges and that the resulting matching is stable. It is easy to see that $f(\mathcal{X},\varnothing,\mu^*,\varnothing) \sqsubseteq (\mathcal{X},\varnothing,\mu^*,\varnothing)$ because $C^{\mathcal{S}}(\mathcal{X}|\mu^*) \precsim^{\mathcal{S}} \mu^*$ by construction and $C^{\mathcal{B}}(\varnothing|\varnothing) = \varnothing \succsim^{\mathcal{B}} \varnothing$ by reflexivity of $\succsim^{\mathcal{B}}$. By Lemma 2, f is monotone increasing, so we can repeatedly apply it to $f(\mathcal{X},\varnothing,\mu^*,\varnothing) \sqsubseteq (\mathcal{X},\varnothing,\mu^*,\varnothing)$ to get $f^k(\mathcal{X},\varnothing,\mu^*,\varnothing) \sqsubseteq f^{k-1}(\mathcal{X},\varnothing,\mu^*,\varnothing)$ for every $k \geq 1$. We consider two separate possibilities. Suppose first that this sequence converges. Therefore, there exists k such that $f^{k-1}(\mathcal{X},\varnothing,\mu^*,\varnothing) = f^k(\mathcal{X},\varnothing,\mu^*,\varnothing)$. As a result, $f^{k-1}(\mathcal{X},\varnothing,\mu^*,\varnothing)$ is a fixed point of f. Let $(\hat{A}^s,\hat{A}^b,\hat{\mu}^s,\hat{\mu}^b) \equiv f^{k-1}(\mathcal{X},\varnothing,\mu^*,\varnothing)$. By Lemma 3, $\hat{\mu}^s = \hat{\mu}^b = \hat{A}^s \cap \hat{A}^b$ and, by Theorem 7, $\hat{A}^s \cap \hat{A}^b$ is a stable matching.

Otherwise, if the sequence does not converge, there exists a subsequence $f^n(\mathcal{X},\varnothing,\mu^*,\varnothing) \sqsupseteq f^{n+1}(\mathcal{X},\varnothing,\mu^*,\varnothing) \sqsupseteq \ldots \sqsupseteq f^m(\mathcal{X},\varnothing,\mu^*,\varnothing) \sqsupseteq f^{m+1}(\mathcal{X},\varnothing,\mu^*,\varnothing) = f^n(\mathcal{X},\varnothing,\mu^*,\varnothing)$ because the number of contracts is finite. By transitivity of the preorder $\sqsupseteq$ and the previous inequality, we get $f^n(\mathcal{X},\varnothing,\mu^*,\varnothing) = f^{m+1}(\mathcal{X},\varnothing,\mu^*,\varnothing) \sqsupseteq f^m(\mathcal{X},\varnothing,\mu^*,\varnothing) \sqsupseteq f^n(\mathcal{X},\varnothing,\mu^*,\varnothing)$. Let $f^n(\mathcal{X},\varnothing,\mu^*,\varnothing) = (A_1^s,A_1^b,\mu_1^s,\mu_1^b)$ and $f^m(\mathcal{X},\varnothing,\mu^*,\varnothing) = (A_2^s,A_2^b,\mu_2^s,\mu_2^b)$. By definition of $\sqsupseteq$, we get that $A_1^s = A_2^s$, $A_1^b = A_2^b$, $\mu_1^s \sim^s \mu_2^s$, and $\mu_1^b \sim^b \mu_2^b$. Now, by construction $C^{\mathcal{S}}(A_2^s|\mu_2^s) = \mu_1^s$ and by monotone externalities $C^{\mathcal{S}}(A_2^s|\mu_2^s) = C^{\mathcal{S}}(A_1^s|\mu_1^s)$, which imply that $C^{\mathcal{S}}(A_1^s|\mu_1^s) = \mu_1^s$. Similarly, we get that $C^{\mathcal{S}}(A_1^s|\mu_1^b) = \mu_1^b$. Furthermore, by monotone externalities, $\mathcal{X}\backslash R^{\mathcal{B}}(A_2^b|\mu_2^b) = \mathcal{X}\backslash R^{\mathcal{B}}(A_1^b|\mu_1^b)$ and, by construction, $\mathcal{X}\backslash R^{\mathcal{B}}(A_2^b|\mu_2^b) = A_1^b$, which imply $\mathcal{X}\backslash R^{\mathcal{B}}(A_1^b|\mu_1^b) = A_1^b$. Similarly, we get $\mathcal{X}\backslash R^{\mathcal{S}}(A_1^s|\mu_1^s) = A_1^s$. Therefore, $(A_1^s,A_1^b,\mu_1^s,\mu_1^b)$ is a fixed point of f. This shows that the sequence converges as in the previous paragraph, which is a contradiction. Therefore, there exists a stable matching. ∎

Proof of Theorem 2

Since choice function c_i has externalities, there exist $X,\mu,\mu' \subseteq \mathcal{X}$ such that $c_i(X|\mu') \neq c_i(X|\mu)$. This implies, without loss of generality, that there exists a contract $x \in X_i$ such that $x \in c_i(X_i|\mu'_{-i})$ and $x \notin c_i(X_i|\mu_{-i})$. We construct choice functions of agents other than i satisfying the stated properties such that no stable matching exists.

The choice functions of agents on side $-\theta$ exhibit no externalities. Furthermore, each agent chooses all the contracts in $\mu_{-i} \cup \mu'_{-i} \cup X_i$ that are associated with them whenever they are available. No other contracts are chosen. The choice functions of agents on side θ other than i depend on whether the reference set has contract x or not. When contract x is in the reference set, each agent chooses contracts in μ_{-i} associated with them. When contract x is not in the reference set, then each agent chooses contracts in μ'_{-i} associated with them. Otherwise, no contracts are chosen.

We first check that the properties in the statement of this result are satisfied. The agents on side $-\theta$ have choice functions that have no externalities. Furthermore, $C^{-\theta}$ satisfies substitutability and the irrelevance of rejected contracts. Now, consider the minimum consistent preorder $\gtrsim^{\theta\setminus\{i\}}$ for $C^{\theta\setminus\{i\}}$. Any reference set μ in the domain of $\gtrsim^{\theta\setminus\{i\}}$ does not include contract x because, for every agent $j \in \theta \setminus \{i\}$, $\gtrsim_j$ is a preorder with a domain that is a subset of 2^{X_j}, so no matching in this domain includes contract x. Therefore, for any $X \subseteq \mathcal{X}$, $C^{\theta\setminus\{i\}}(X|\mu)$ is the same for all μ in the domain of $\gtrsim^{\theta\setminus\{i\}}$ because μ does not have contract x, implying that monotone externalities is satisfied. Furthermore, by construction, standard substitutability and the irrelevance of rejected contracts are also satisfied. Hence, $C^{\theta\setminus\{i\}}$ satisfies substitutability.

Suppose, for contradiction, that there exists a stable matching Y. We consider two possibilities:

Case 1: Consider the case when $x \in Y$. If a contract in μ_{-i} is not in Y, then the agents associated with the contract form a blocking pair. Thus, every contract in μ_{-i} must be signed, so $\mu_{-i} \subseteq Y_{-i}$. Furthermore, $Y_{-i} \setminus \mu_{-i}$ cannot have a contract as Y would not be individually rational for agents on side θ. Therefore, $\mu_{-i} = Y_{-i}$. Likewise, there cannot be any contract in $Y_i \setminus X_i$ because of individual rationality for agents on side $-\theta$. This implies that $Y_i \subseteq X_i$. If there exists a contract $x' \in c_i(X_i|\mu_{-i}) \setminus Y_i$, then agents associated with contract x' block Y because $x' \in c_i(Y_i \cup \{x'\}|\mu_{-i})$ by standard substitutability. Therefore, $Y_i \supseteq c_i(X_i|\mu_{-i})$. By the irrelevance of rejected contracts, $c_i(Y_i|\mu_{-i}) = c_i(X_i|\mu_{-i})$, which is a contradiction since $x \in Y_i = c_i(Y_i|\mu_{-i})$ by individual rationality of Y and $x \notin c_i(X_i|\mu_{-i})$ by construction.

Case 2: Consider the case when $x \notin Y$. As in the previous case, it is easy to see that $Y_{-i} = \mu'_{-i}$. Likewise, $Y_i \subseteq X_i$. Since $x \in c_i(X_i|\mu'_{-i})$ by construction, $x \in c_i(Y_i \cup \{x\}|\mu'_{-i})$ by standard substitutability. But this is a contradiction because $x \notin Y$ implies that the agents associated with contract x form a blocking pair.

Therefore, there exists no stable matching. ■

Proof of Theorem 4

Without loss of generality assume that $\theta = \mathcal{S}$. For any $(A^s, A^b, \mu^s, \mu^b) \in 2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}}$ we have $(\mathcal{X}, \emptyset, \bar{\mu}^s, \emptyset) \sqsupseteq (A^s, A^b, \mu^s, \mu^b)$. Therefore, $(\mathcal{X}, \emptyset, \bar{\mu}^s, \emptyset) \sqsupseteq f(\mathcal{X}, \emptyset, \bar{\mu}^s, \emptyset)$. By Lemma 2, function f is monotone increasing, so we can repeatedly apply it to the last inequality to get $f^{k-1}(\mathcal{X}, \emptyset, \bar{\mu}^s, \emptyset) \sqsupseteq f^k(\mathcal{X}, \emptyset, \bar{\mu}^s, \emptyset)$ for every $k \geq 1$. Since $2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}}$ is a finite set, this sequence converges at some point as in the proof of Theorem 1, so there exists k such that $f^{k-1}(\mathcal{X}, \emptyset, \bar{\mu}^s, \emptyset) = f^k(\mathcal{X}, \emptyset, \bar{\mu}^s, \emptyset)$. Therefore, $f^{k-1}(\mathcal{X}, \emptyset, \bar{\mu}^s, \emptyset)$ is a fixed point of f. By Lemma 3 there is $\left(\hat{A}^s, \hat{A}^b, \hat{\mu}, \hat{\mu}\right)$ that is equal to $f^{k-1}(\mathcal{X}, \emptyset, \bar{\mu}^s, \emptyset)$. Theorem 7 tells us that $\hat{\mu}$ is a stable matching, which is the outcome of the seller-proposing deferred-acceptance algorithm.

We next show that $\hat{\mu}$ is a seller-optimal and buyer-pessimal stable matching. Let μ be any stable matching. By Theorem 7, there exist A^s and A^b such that (A^s, A^b, μ, μ) is a fixed point of f. Since $(\mathcal{X}, \emptyset, \bar{\mu}^s, \emptyset) \sqsupseteq (A^s, A^b, \mu, \mu)$ and f is monotonic increasing, f can be applied repeatedly while preserving the order. Therefore, $f^k(\mathcal{X}, \emptyset, \bar{\mu}^s, \emptyset) \sqsupseteq f^k(A^s, A^b, \mu, \mu)$ for every k, which implies $\left(\hat{A}^s, \hat{A}^b, \hat{\mu}, \hat{\mu}\right) \sqsupseteq (A^s, A^b, \mu, \mu)$. Therefore, $\hat{\mu} \succsim^{\mathcal{S}} \mu$ and $\hat{\mu} \precsim^{\mathcal{B}} \mu$, so $\hat{\mu}$ is a seller-optimal and buyer-pessimal stable matching. ■

Proof of Theorem 5

We first show the necessity that when C^θ satisfies substitutability, then, for each agent $i \in \theta$, there exists a list of preferences with the stated properties.

For any μ_{-i}, we can construct a list of preferences as follows. Let $x_1 \in c_i(\mathcal{X}|\mu_{-i})$, $x_2 \in c_i(\mathcal{X} \setminus \{x_1\}|\mu_{-i})$, $x_3 \in c_i(\mathcal{X} \setminus \{x_1, x_2\}|\mu_{-i})$, $\ldots$, $x_k \in c_i(\mathcal{X} \setminus \{x_1, \ldots, x_{k-1}\}|\mu_{-i})$, and $c_i(\mathcal{X} \setminus \{x_1, \ldots, x_k\}|\mu_{-i}) = \emptyset$. This sequence creates an incomplete preference ranking over $\mathcal{X}_i \cup \{\emptyset\}$: $x_1 \succ^{\mu_{-i}} \ldots \succ^{\mu_{-i}} x_k \succ^{\mu_{-i}} \emptyset$. Consider all such preference rankings $(\succ_j^{\mu_{-i}})_{j \in \mathcal{J}}$. We need the following:

Claim: For any $X, \mu \subseteq \mathcal{X}$, $c_i(X|\mu_{-i}) = \bigcup_{j \in \mathcal{J}} \{x_j^{\mu_{-i}}\}$, where $x_j^{\mu_{-i}} = \max_{\succ_j^{\mu_{-i}}} (X \cup \{\emptyset\})$.[35]

Let $x \in c_i(X|\mu_{-i})$. We show that $x = x_j^{\mu_{-i}}$ for some $j \in \mathcal{J}$ when X is the set of contracts. If $x \in c_i(\mathcal{X}|\mu_{-i})$, then $x = x_j^{\mu_{-i}}$ for some j. Suppose that $x \notin c_i(\mathcal{X}|\mu_{-i})$. If $c_i(X|\mu_{-i}) \supseteq c_i(\mathcal{X}|\mu_{-i})$, then the irrelevance of rejected contracts would imply $c_i(X|\mu_{-i}) = c_i(\mathcal{X}|\mu_{-i})$, which is a contradiction because $x \in c_i(X|\mu_{-i}) \setminus c_i(\mathcal{X}|\mu_{-i})$. Therefore, there exists $x_1 \in c_i(\mathcal{X}|\mu_{-i}) \setminus c_i(X|\mu_{-i})$. Standard substitutability implies that $x_1 \notin X$. Consider preference rankings in $\mathcal{J}$ that have x_1 as their maximal contract. If $x \in c_i(\mathcal{X} \setminus \{x_1\}|\mu_{-i})$, then we are done since x_1 would be the

[35] For an analogue of this claim in the setting without externalities, see Chambers and Yenmez (2017).

maximal element of X with respect to a preference ranking since $x_1 \notin X$ and there would be a preference ranking in $\mathcal{J}$ such that $x_1 \succ x \succ \ldots$. Suppose that $x \notin c_i(\mathcal{X} \setminus \{x_1\}|\mu_{-i})$. By the irrelevance of rejected contracts, we cannot have $c_i(X|\mu_{-i}) \supseteq c_i(\mathcal{X} \setminus \{x_1\}|\mu_{-i})$. Therefore, there exists $x_2 \in c_i(\mathcal{X} \setminus \{x_1\}|\mu_{-i}) \setminus c_i(X|\mu_{-i})$. Standard substitutability implies that $x_2 \notin X$. Repeat this argument. Suppose, for contradiction, that $x \notin c_i(\mathcal{X} \setminus \{x_1, \ldots, x_j\}|\mu_{-i})$ for all j. But there must exist some j^* for which $\mathcal{X} \setminus \{x_1, \ldots, x_{j^*}\} \subseteq X$. Then $x \in c_i(X|\mu_{-i})$ and standard substitutability imply that $x \in c_i(\mathcal{X} \setminus \{x_1, \ldots, x_{j^*}\}|\mu_{-i})$. This is a contradiction. Therefore, $x \in c_i(\mathcal{X} \setminus \{x_1, \ldots, x_{j^*}\}|\mu_{-i})$ for some j^*, which implies that $x = x_j^{\mu_{-i}}$ for some $j \in \mathcal{J}$ because $\{x_1, \ldots, x_{j^*}\} \cap X = \emptyset$. Since $x \in c_i(X|\mu_{-i})$ implies $x = x_j^{\mu_{-i}}$ for some $j \in \mathcal{J}$, we get $c_i(X|\mu_{-i}) \subseteq \bigcup_{j \in \mathcal{J}} \{x_j^{\mu_{-i}}\}$.

Now let $x = x_j^{\mu_{-i}}$ for some j. This implies that for every $y \succ_j^{\mu_{-i}} x$, we have $y \notin X$. By construction, $x \in c_i(\mathcal{X} \setminus \bigcup_{y: y \succ_j^{\mu_{-i}} x} \{y\}|\mu_{-i})$. Standard substitutability and the fact that $\mathcal{X} \setminus \bigcup_{y: y \succ_j^{\mu_{-i}} x} \{y\} \supseteq X$ imply that $x \in c_i(X|\mu_{-i})$. This argument proves that $\bigcup_{j \in \mathcal{J}} \{x_j^{\mu_{-i}}\} \subseteq c_i(X|\mu_{-i})$. Therefore, $\bigcup_{j \in \mathcal{J}} \{x_j^{\mu_{-i}}\} = c_i(X|\mu_{-i})$, which concludes the proof of the claim.

Next we prove that, for any $\mu'_{-i} \succsim^\theta \mu_{-i} \succsim^\theta \emptyset$ and $j \in \mathcal{J}$, $\succ_j^{\mu'_{-i}}$ is a truncation of $\succ_j^{\mu_{-i}}$.

Take $\mu = \emptyset$ and construct the list of preferences $(\succ_j^{\emptyset})_{j \in \mathcal{J}}$ as above. For any $\mu_{-i} \succsim^\theta \emptyset$ and $X \subseteq \mathcal{X}$, $c_i(X|\mu_{-i}) \subseteq c_i(X|\emptyset)$ by monotone externalities. Thus, for each j, we can truncate the preference ranking $\succ_j^{\emptyset}$ to get a sequence as constructed above, call it $\succ_j^{\mu_{-i}}$.

For each $\mu{-}i \succsim^\theta \emptyset$, $c_i(X|\mu_{-i}) = \bigcup_{j \in \mathcal{J}} \{x_j^{\mu_{-i}}\}$ where $x_j^{\mu_{-i}} = \max_{\succ_j^{\mu_{-i}}} (X \cup \{\emptyset\})$ by construction. Furthermore, for any $\mu'_{-i} \succsim^\theta \mu_{-i} \succsim^\theta \emptyset$ and $X \subseteq \mathcal{X}$, $c_i(X|\mu'_{-i}) \subseteq c_i(X|\mu_{-i})$ by monotone externalities. Therefore, for any j, $\succ_j^{\mu'_{-i}}$ and $\succ_j^{\mu_{-i}}$ are both truncations of $\succ_j^{\emptyset}$ such that $\succ_j^{\mu'_{-i}}$ is truncated at a weakly more-preferred contract than $\succ_j^{\mu_{-i}}$. Therefore, we get the conclusion that for any $j \in \mathcal{J}$, $\succ_j^{\mu'_{-i}}$ is a truncation of $\succ_j^{\mu_{-i}}$.

Finally, we show the sufficiency that when there exists a list of preferences with the desired properties, then C^θ satisfies substitutability. Standard substitutability follows from the decomposition result of Aizerman and Malishevski (1981). To show monotone externalities, suppose that $\mu' \succsim^\theta \mu \succsim^\theta \emptyset$, we need $R^\theta(X|\mu') \supseteq R^\theta(X|\mu)$ for every $X \subseteq \mathcal{X}$. Equivalently, we need that $r_i(X_i|\mu'_{-i}) \supseteq r_i(X_i|\mu_{-i})$ for every $i \in \theta$ and $X \subseteq \mathcal{X}$. By the definition of $\succsim^\theta$, $\mu' \succsim^\theta \mu \succsim^\theta \emptyset$ implies $\mu'_{-i} \succsim^\theta \mu_{-i} \succsim^\theta \emptyset$ for every $i \in \theta$. By construction, there exists a list of preference rankings $(\succ_j^{\mu_{-i}})_{j \in \mathcal{J}}$ and $(\succ_j^{\mu'_{-i}})_{j \in \mathcal{J}}$ such that for every $j \in \mathcal{J}$, $\succ_j^{\mu'_{-i}}$ is a truncation of $\succ_j^{\mu_{-i}}$. Therefore, $r_i(X_i|\mu'_{-i}) \supseteq r_i(X_i|\mu_{-i})$ is satisfied. ■

Appendix E: Couple in Local Labor Market: An Extension

We can generalize the couples application in Section 3 so that there are externalities for both partners in a couple. For each individual in a married couple the set of jobs are divided into three sets. The first set has the most preferred “dream jobs.” The second set has less preferred “decent jobs.” The last set has the least preferred “unacceptable jobs.” Dream jobs are always more preferred than the outside option. Unacceptable jobs are always less preferred than the outside option. Unlike these two sets of jobs, a decent job is sometimes more preferred than the outside option and sometimes less preferred depending on the spouse’s job: when the spouse has a decent job than all decent jobs are more preferred than the outside option, whereas when the spouse has a dream job some of the decent jobs are less preferred than the outside option. The pairwise ranking of jobs remains the same regardless of the spouse’s job.

In this more general version of the couples’ application, consider the following preorder for married individuals. For each married worker i there is a primitive ranking of jobs, which can be based on the wages, and the outside option of being unemployed, say $\succeq_i$. Then define the preorder $\gtrsim_i$ so that $j' \gtrsim_i j$ if $j' \succeq_i j$, or j' and j are both decent jobs, or j' and j are both outside options. In particular, all decent jobs are ranked as equivalent by $\gtrsim_i$. The resulting preorder is consistent because as there are more jobs available regardless of the reference sets, every married individual i gets a weakly more preferred job with respect to $\gtrsim_i$. Substitutability is satisfied because a married individual becomes weakly more selective whenever their spouse gets a more preferred job, so they reject weakly more jobs conditional on μ' compared to μ whenever $\mu' \gtrsim^\theta \mu$. ■

Appendix F: Additional Applications

In this section, we provide additional applications that satisfy substitutability. With the exception of our characterization result, which is Theorem 5, all our results work even when the preorder for a side is not necessarily defined using preorders of agents on this side.[36] Some of the applications below allow for this generality.

Application 2. [Relative Rankings in Hiring] Agents on one side of the market represent colleges and agents on the other side represent academics in a particular field. For each college

[36]See the previous version of our paper, which is available at `http://dx.doi.org/10.2139/ssrn.2475468`.

i and each academic j the productivity of j at i is denoted by $\lambda(i, j) \geq 0$. For simplicity, assume that no two academics have the same productivity at a college.[37]

Suppose that each college i hires at most two academics in the field considered, and that it wants to hire at least one because of teaching needs and would like to hire a second academic only if their productivity is weakly greater than a benchmark that depends on the productivity of hires at other colleges. Formally, the choice function $c_i(X_i|\mu)$ of college i is as follows: from choice set X_i, the college chooses the academic $j \in X_i$ with highest productivity $\lambda(i, j)$, and it chooses a second academic $j' \in X_i$ with second-highest productivity in X_i if, and only if, $\lambda(i, j') \geq b_i(\mu)$ where $b_i(\mu)$ is a benchmark productivity of academics at other colleges. We assume that $b_i(\mu)$ is weakly increasing in $\max_{j \in \mu(i')} \lambda(i', j)$ for all colleges $i' \neq i$. For instance, $b_i(\mu)$ might equal the median productivity of the leading academic in other colleges, where j is the leading academic in college i' if $j = \arg\max_{j \in \mu(i')} \lambda(i', j)$). Or, $b_i(\mu)$ might be equal to other percentiles of leading academics' productivity distribution. The interpretation is that a second academic is hired only if they are a "star" in the field.

College choice functions satisfy substitutability if we define the preorder $\succsim^\theta$ so that for each college i, $\mu' \succsim^i \mu$ if, and only if, $\max_{j \in \mu'(i)} \lambda(i, j)$ is weakly greater than $\max_{j \in \mu(i)} \lambda(i, j)$.[38] This preorder is consistent with the choice functions: when more academics are available then the maximum quality of the academics a college hires goes up (whether or not the benchmark quality of academics increases). The substitutability condition is then satisfied: when more academics are available and when the benchmark quality of academics increases, each college continues to reject the academics it previously rejected. ■

Application 3. [Dynamic Matching][39] Firms and workers arrive to a two-sided matching market at times $t = 1, \ldots, T$. Workers who arrive at time t can wait and match at any time $t, t+1, \ldots, T$. At each time t a unique firm f_t arrives and either matches with one of the workers that is available at this time, or leaves unmatched. Firm f_t's ranking of workers is exogenously fixed but this firm's set of acceptable workers depends on the matches of firms $f_1, \ldots, f_{t-1}$: the higher firm f_1's worker in f_1's ranking, the more selective firm f_t becomes. If firm f_1 hires the same worker in two matchings, then the higher firm f_2's worker in f_2's ranking, the more selective firm f_t becomes, etc., lexicographically.

In this application, a consistent preorder for the firms is defined as follows: $\mu' \succsim^\theta \mu$ if, and only if, for some firm f we have $\mu'(f) \succ_f \mu(f)$ and $\mu'(f') \succsim_{f'} \mu(f')$ for all firms f'

[37]For concreteness, we are using the academic job market in this application but this could be any job market.
[38]When $\mu(i)$ is empty, we set the maximum equal to $-\infty$.
[39]We would like to thank Maciej Kotowski for suggesting this application.

matched before f. This preorder is consistent with the choice functions, and the substitutability condition is satisfied as choosing out of larger (in inclusion sense) choice set conditional on a matching higher in this preorder, each firm continues to reject the worker it previously rejected.

■

Our theory applies to situations in which agents share profits, for instance because they work for the same firm, or have some insurance arrangements, or benefit from a public good financed by taxes on their private income. The following application illustrates a situation in which there is profit sharing.

Application 4. **[Profit Sharing]** Agents on one side of the market represent attorneys organized in law firms. Each attorney can work on up to $k \geq 0$ contracts with clients on the other side of the market; an attorney works on all contracts they sign and the attorney can also work on selected contracts signed by others in the same firm. Each contract allows an arbitrary number of attorneys to contribute; the profit an attorney makes from a contract does not depend on how many other attorneys contribute to it.[40] Each attorney prioritizes the contracts they work on, and the profit attorney i earns on a contract depends on whether it is the first, second, etc. contract in attorney i's priorities. We assume that each attorney must prioritize the contracts they sign over other contracts that they work on.

Attorneys choose what contracts to sign and what contracts to work on so as to maximize their profits: An attorney's profit is the sum of the profits from all the contracts they work on whether they signed it or not. We denote by $\lambda(x, i, \ell) \geq 0$ the profit that accrues to attorney i from working on contract x that they prioritize in position $\ell \in \{1, \ldots, k\}$. For simplicity, let us also assume that there are no indifferences. This application satisfies our assumptions provided $\lambda(x, i, 1) > \lambda(y, i, \ell)$ for all contracts x and y as long as attorney i is the signatory of contract x and $\ell > 1$.

Attorney choice functions satisfy substitutability if we define the preorder $\gtrsim^\theta$ so that $\mu' \gtrsim^\theta \mu$ if, and only if, $\max_{x \in \mu'(i)} \lambda(x, i, 1) \geq \max_{x \in \mu(i)} \lambda(x, i, 1)$ for all agents $i \in \theta$.[41] This preorder is consistent with choice: When more contracts are available, the profitability of the best contract signed by each attorney goes up (irrespective of what contracts other attorneys sign). The substitutability condition holds for each attorney i: When more contracts are available and when the profitability of the best contract signed by other attorneys (and hence the outside option of attorney i) increases, the attorney continues to reject the contracts they previously

[40]This assumption and some of our other assumptions can be relaxed.

[41]We use the convention that the maximum over the empty set is $-\infty$.

rejected. ■

Our theory also applies to situations in which agents choose basic products with no regard to the choices of others but choose add-ons in a way that depends on others' choices of basic products. For instance, consider buyers who choose between Mac, PC, and Linux computers (and operating systems) in a way that does not depend on other buyers' choices and who take the hardware/operating system choices of others into account when buying productivity software.

Application 5. [Interoperability and Add-on Contracts] Suppose agents on one side (buyers) sign two types of contracts with sellers on the other side: for instance, agents might be signing primary contracts and add-on (or maintenance) contracts. These two classes of contracts are disjoint.[42] In line with the literature on add-on pricing, suppose that agents ignore the add-on contracts when deciding which primary contracts to sign (Gabaix and Laibson, 2006), and suppose that each agent signs at most one primary contract and that there are no externalities among primary contracts.[43]

We assume that no agent's choice of add-on contracts depends on the other agents' choices of add-on contracts, and we allow a buyer's choice among add-on contracts to depend on their and the other agents' choices of primary contracts in an arbitrary way as long as the buyer rejects weakly more (in the inclusion sense) add-on contracts out of X conditional on μ than they would reject out of X' conditional on μ' whenever $X \supseteq X'$ and the agent prefers their primary contracts in μ to those in μ'.

Buyer choice functions satisfy substitutability for the preorder $\gtrsim^\theta$ such that $\mu' \gtrsim^\theta \mu$ when each buyer prefers their primary contracts signed under μ' to those signed under μ. This preorder is consistent: $\gtrsim^\theta$ depends only on primary contracts, and each agent prefers to choose from larger choice sets over choosing from smaller choice sets. It is enough to check substitutability separately for the primary contracts and the add-on contracts: it holds for the primary contracts as the choice over them is not affected by externalities, and it holds for the add-on contracts as we explicitly assumed it. ■

[42]Similar applications can be written for hardware contracts and software contracts, or contracts on inputs and outputs.

[43]Formally, we assume that each buyer's choice among primary contracts does not depend on other agents' matches nor on the availability of add-on contracts. One reason that the agents ignore add-on contracts when signing primary contracts might be that the agents do not know which add-on contracts are available when signing the primary contracts as in Ellison (2005). We can relax the assumption that each agent signs at most one primary contract and assume instead that each agent's choice among primary contracts satisfies the standard substitutes assumption (see the next section).

Appendix G: Comparative Statics

How do stable matchings change when agents' choice functions stop (or begin) exhibiting externalities? We answer this question controlling for the agents' propensity to reject contracts.[44]

Definition 7. Choice function C^θ is an **expansion** of choice function $\hat{C}^\theta$ if, for any $\mu, X \subseteq \mathcal{X}$,

$$C^\theta(X|\mu) \supseteq \hat{C}^\theta(X|\mu).$$

We then also say that $\hat{C}^\theta$ is a **contraction** of C^θ.

In words, when choice function C^θ is an expansion of choice function $\hat{C}^\theta$, it admits weakly more contracts (in the superset sense) than $\hat{C}^\theta$ for any set of available contracts and reference set. Likewise, a contraction of a choice function selects weakly less contracts for any set of contracts and reference set. A natural instance of contraction is when contracts are substitutes under both $\hat{C}^\theta$ and C^θ and contracts are *closer substitutes* under $\hat{C}^\theta$ than under C^θ: the strength of substitutability among two contracts being measured by whether an agent is willing to choose both of them or not. For instance, in relative ranking in hiring example in Appendix F, when a college has larger k, which is the share of other colleges it benchmarks itself against, it becomes more reluctant to hire more than one academic making academics closer substitutes for this college.

Controlling for the agents' propensity to reject contracts allows us to establish unambiguous comparative statics: removing externalities while contracting choice for one side of the market benefits this side and harms the other side.

Theorem 8. *(**Comparative Statics**) Suppose that the choice functions $C^\mathcal{B}$, $C^\mathcal{S}$, and $C^{*\mathcal{S}}$ satisfy substitutability, $C^\mathcal{S}$ does not exhibit externalities and it is a contraction of $C^{*\mathcal{S}}$. Then, for any $\left(C^\mathcal{B}, C^{*\mathcal{S}}\right)$-stable matching μ^* there exists a $\left(C^\mathcal{B}, C^\mathcal{S}\right)$-stable matching μ such that*

$$\mu \gtrsim^\mathcal{S} \mu^* \ \ \textit{and} \ \ \mu^* \gtrsim^\mathcal{B} \mu,$$

where $\gtrsim^\mathcal{S}$ is the Blair order for $C^\mathcal{S}$ and $\gtrsim^\mathcal{B}$ is a consistent preorder for $C^\mathcal{B}$.

[44]The 2014-2019 drafts of our paper developed the comparative statics for both the case with and without externalities. We are now developing the no-externalities case as an independent paper and the marginal contribution of the present discussion to extend the results to the case with externalities; we thank a referee for the suggestion to split off the no-externality results. At the same time we developed our analysis, related issues (for the no-externality case) were also studied by Echenique and Yenmez (2015) and Chambers and Yenmez (2017) who introduced the terminology of choice function C^θ being an expansion of choice function $\hat{C}^\theta$ while we originally used the terminology of C^θ exhibiting weaker substitutes than $\hat{C}^\theta$; cf. also Kamada and Kojima (2020).

One application of this result is to the couples in local labor markets setting of Section 3, in which there are externalities among members of a couple, while firms' choices do not exhibit externalities. Suppose that two workers get married. The marriage contracts the preferences of the (post-marriage) secondary earner while not changing the preferences of the (post-marriage) primary earner. Theorem 8—with workers playing the role of sellers of labor and firms playing the role of buyers—then implies that for any for any matching μ^* that was stable before the marriage there exists a matching μ that is stable post marriage such that $\mu \succsim^{\mathcal{S}} \mu^*$ and $\mu^* \succsim^{\mathcal{B}} \mu$. This means in our context that all firms prefer the job matching before the marriage while all primary earners prefer the job matching post marriage.

Proof of Theorem 8. Since $\succsim^{\mathcal{S}}$ is the Blair order for substitutable choice function $C^{\mathcal{S}}$—which does not exhibit externalities—we have

$$C^{\mathcal{S}}(X|\mu) \succsim^{\mathcal{S}} C^{*\mathcal{S}}(X|\mu)$$

for any $\mu, X \subseteq \mathcal{X}$.[45] Because μ^* is a $(C^{\mathcal{B}}, C^{*\mathcal{S}})$-stable matching, Theorem 7 gives us sets $A^s, A^b \subseteq \mathcal{X}$ such that (A^s, A^b, μ^*, μ^*) is a fixed point of the $\left(C^{\mathcal{B}}, C^{*\mathcal{S}}\right)$-analogue of function f from Lemma 2, defined as

$$\hat{f}\left(A^s, A^b, \mu^s, \mu^b\right) \equiv \left(\mathcal{X}\backslash R^{\mathcal{B}}(A^b|\mu^b), \mathcal{X}\backslash R^{*\mathcal{S}}(A^s|\mu^s), C^{*\mathcal{S}}(A^s|\mu^s), C^{\mathcal{B}}(A^b|\mu^b)\right).$$

The fixed point property, the contraction relation, and the above displayed property of $\succsim^{\mathcal{S}}$ imply that

$$(A^s, A^b, \mu^*, \mu^*) \sqsubseteq f(A^s, A^b, \mu^*, \mu^*),$$

where mapping f and preorder $\sqsubseteq$ are defined in Lemma 2. Indeed,

$$A^s = \mathcal{X}\backslash R^{\mathcal{B}}(A^b|\mu^*)$$

by the fixed point property;

$$A^b = \mathcal{X}\backslash R^{*\mathcal{S}}(A^s|\mu^*) \supseteq \mathcal{X}\backslash R^{\mathcal{S}}(A^s|\mu^*)$$

[45] The argument below applies also to any $C^{\mathcal{S}}$ with externalities as long as it admits a consistent preorder that satisfies the displayed property. As with the substitutes comparison, we can further weaken this property by imposing it only when $C^{\theta}(X|\mu) = \mu$; the weaker assumptions suffice as in the proof we apply this property to $C^{\mathcal{S}}$ and $\hat{C}^{\mathcal{S}}$ only when $C^{\mathcal{S}}(A^s|\mu) = \mu$.

by the fixed point property and the contraction relation between $C^{\mathcal{S}}$ and $C^{*\mathcal{S}}$,

$$\mu^* = C^{*\mathcal{S}}(A^s|\mu^*) \precsim^{\mathcal{S}} C^{\mathcal{S}}(A^s|\mu^*) \tag{1}$$

by the fixed point property and the above displayed property of $\precsim^{\mathcal{S}}$;

$$\mu^* \succsim^{\mathcal{B}} \mu^* = C^{\mathcal{B}}(A^b|\mu^*) \tag{2}$$

by the fixed point property.

By Lemma 2, f is monotone increasing in preorder $\sqsubseteq$ and $f^{\ell-1}(A^s, A^b, \mu^*, \mu^*) \sqsubseteq f^{\ell}(A^s, A^b, \mu^*, \mu^*)$ for every $\ell \geq 1$. We denote by μ^s_ℓ and μ^b_ℓ the reference matchings of sellers and buyers (respectively) in $f^{\ell}(A^s, A^b, \mu^*, \mu^*)$. Since the number of contracts is finite, there exists $k \geq 1$ such that $f^{k-1}(A^s, A^b, \mu^*, \mu^*)$ is a fixed point of f as in the proof of Theorem 1. By Lemma 3, there are contract sets $\hat{A}^s, \hat{A}^b$ such that $f^{k-1}(A^s, A^b, \mu^*, \mu^*) = (\hat{A}^s, \hat{A}^b, \mu^s_{k-1}, \mu^b_{k-1})$ and $\mu^s_{k-1} = \mu^b_{k-1}$. Denoting this common reference matching by μ, we infer from Theorem 7 that μ is a $(C^{\mathcal{B}}, C^{\mathcal{S}})$-stable matching. By (1) and (2) and the monotonicity of f, at every step of iterating f we have $\mu^s_{\ell-1} \succsim^{\mathcal{S}} \mu^*_\ell$ and $\mu_\ell \succsim^{\mathcal{B}} \mu_{\ell-1}$; hence $\mu \succsim^{\mathcal{S}} \mu^*$ and $\mu^* \succsim^{\mathcal{B}} \mu$. ■