Auctions of Homogeneous Goods: A Case for Pay-as-Bid

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Abstract

The pay-as-bid (or discriminatory) auction is a prominent format for selling homogenous goods such as treasury securities and commodities. We prove the uniqueness of its pure-strategy Bayesian Nash equilibrium and establish a tractable representation of equilibrium bids. Building on these results we analyze the optimal design of pay-as-bid auctions, as well as uniform-price auctions (the main alternative auction format), allowing for asymmetric information. We show that supply transparency and full disclosure are optimal in pay-as-bid, though not necessarily in uniform-price; pay-as-bid is revenue dominant and might be welfare dominant; and, under assumptions commonly imposed in empirical work, the two formats are revenue and welfare equivalent.

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1 Introduction

Each year, securities and commodities worth trillions of dollars are allocated through multi-unit auctions. The two primary auction formats for these sales are pay-as-bid and uniform-price. Pay-as-bid is the more popular of the two auction formats for selling treasury securities, and it is frequently implemented to distribute electricity generation. It is also used in other government operations, including recent large-scale asset purchases in the U.S. (quantitative easing), and is implicitly run in financial markets when limit orders are followed by a market order.[1] Despite their economic importance, relatively little is known about equilibrium behavior in pay-as-bid auctions. Accordingly, little is known about the design problem faced by the pay-as-bid auctioneer: for instance, what is the optimal reserve price, and how does transparency about supply affect the seller's revenue? Furthermore, empirical studies find rough revenue equivalence of pay-as-bid and uniform-price auctions, posing an intriguing puzzle for theoretical research.[2]

This paper addresses these open questions. We derive an equilibrium in bidding strategies, conditional on the auction's design, then we solve the auctioneer's problem, taking subsequent bidding equilibria as given. Our first contribution lies in expanding the bounds within which bidding behavior in the pay-as-bid auction is tractable: we allow an arbitrary number of bidders and general demands. We derive bounds on the market-clearing price allowing any asymmetric and asymmetrically-informed bidders. In analyzing further particulars of the bidding equilibrium we initially focus on the case where bidders are symmetrically informed, and we leverage these results to obtain insights that are valid in the presence of informational asymmetries.[3]

[1]Pay-as-bid auctions are also referred to as discriminatory, or multiple-price auctions. Brenner et al. [2009] find that 33 of 48 countries surveyed allocate securities via pay-as-bid auction; Del Río [2017] finds that 27 of 31 markets surveyed distribute electricity generation via pay-as-bid auction (see also Maurer and Barroso [2011]). In both settings, most of the remaining markets are cleared by uniform-price auction. For financial markets, see, e.g., Glosten [1994].

[2]Pay-as-bid auction equilibria have been constructed in parameterized environments; see our discussion below. The empirical literature on multi-unit auctions provides no definitive result on which auction format raises more revenue; Hortaçsu et al. [2018] show that this is potentially because bidders retain little surplus.

[3]Our results are tightest when informational asymmetries among bidders are small, a property satisfied in some important environments. For instance, any issue of treasury securities has both close substitutes whose prices are known, and the forward contracts based on the issue are traded ahead of the auction in the forward markets, thus providing bidders with substantial information about each others' valuations. In empirical analyses, Hortaçsu et al. [2018] argue that bidders in U.S. Treasury auctions of short-term securities are nearly symmetrically informed, and Armantier and Lafhel [2009] show that bidders in Bank of Canada auctions are essentially symmetric. The tightness of our results is not affected by the extent of informational asymmetry between the seller and the bidders; the difference between seller's and bidders' information is typical of the problem we study because the seller designs the auction before (usually substantially before) the bidders submit their bids. We allow for uncertainty of the total supply available for auction; exogenous supply uncertainty is a feature of some securities auctions.

Our theory of equilibrium bidding in pay-as-bid auctions focuses on pure-strategy equilibria; our design insights are valid whether or not we allow mixed strategy-equilibria. For the case with symmetric bidder information, we prove that pure-strategy equilibrium is unique and that bids have an unexpectedly tractable closed-form representation. We also establish a sufficient condition for the existence of equilibrium; our condition is satisfied when, e.g., there are sufficiently many bidders. Going beyond symmetric information, we show that the seller's revenue in an optimally designed pay-as-bid auction with asymmetric information is approximately bounded below by the revenue in the benchmark symmetric-information case.[4]

Our main design result establishes the revenue-optimality of transparently setting supply in pay-as-bid auctions: when bidders have symmetric information, revenue in the unique pure-strategy equilibrium is maximized when supply is deterministic; and, when bidders have small informational asymmetries, revenue is approximately maximized when supply is deterministic. Thus determining the optimal supply distribution is equivalent to solving a standard monopoly problem.[5] Moreover, we show that the seller who cannot design the supply distribution wants to commit to reveal the realization of supply before bids are submitted. That is, it is optimal for the seller to inform bidders of the supply available, regardless of her ability to influence its distribution; this is in sharp contrast with the uniform-price auction, where (we show) deterministic supply is not necessarily revenue-optimal.[6]

We leverage our results on equilibrium bidding and supply transparency in the pay-as-bid auction to compare revenues and welfare in optimally-designed pay-as-bid and uniform-price

[4]We obtain this bound even though it is not clear—nor do we resolve—whether in the limit, as asymmetric information vanishes, asymmetric-information equilibria converge to a pure-strategy equilibrium in the symmetric-information game. The subtlety is not only that different subsequences might in principle converge to different strategy profiles but also that a convergent subsequence might converge to a mixed-strategy profile that is different from the unique pure-strategy equilibrium.

[5]Because the seller in our model can set both a limiting quantity and limiting price, this monopoly problem is not entirely "standard." Nonetheless, it is straightforward to envision a monopolist setting both a limiting price and a limiting quantity. While in this discussion we focus on the seller setting reserve price and distribution of supply, in Appendix A we show that our insights also extend to the case when the seller can set a distribution over elastic supply curves; this extension relies on a Myerson-like regularity assumption imposed on bidders' values.

[6]The reason for this failure is the multiplicity of equilibria in uniform-price auctions. Although there is a uniform-price auction with deterministic supply admitting a revenue-optimal equilibrium, these auctions also admit low- and zero-revenue equilibria (see, e.g., Kremer and Nyborg [2004], LiCalzi and Pavan [2005], McAdams [2007], Burkett and Woodward [2020b], and Marszalec et al. [2020]). Depending on the auctioneer's concern about equilibrium selection, anticipated revenue may improve with some randomization (see also Klemperer and Meyer [1989], as well as our companion analysis of robust uniform-price bidding in Pycia and Woodward [2020]). In practice, in many treasury auctions the distribution of supply is partially determined by the demand from non-competitive bidders, and revenue maximization may not be auctioneer's only objective. However, treasuries and central banks have the ability to influence supply distributions, as well as to release data on non-competitive bids to competitive bidders.

auctions. We prove that the pay-as-bid format always raises weakly higher revenue, while the welfare comparison depends on equilibrium selection in uniform-price auction, which in general allows for multiple equilibria. Approximate revenue dominance and the ambiguity of the welfare comparison remain valid under small asymmetries in bidder information. Major empirical studies comparing revenues between pay-as-bid and uniform-price auctions consider strategy profiles in which bidders in uniform-price bid truthfully for the marginal unit.[7] We show that truthful bidding is one equilibrium of an optimally-designed uniform-price auction and, under this equilibrium selection, we prove that both revenue and welfare are the same across the pay-as-bid and uniform-price auction formats. Thus our results provide a theoretical explanation for the approximate revenue equivalence found by empirical work (see our discussion below).

Before situating our results in the rich related literature, we describe how the pay-as-bid auction operates. First, the bidders submit bids for each infinitesimal unit of the good. Then, the supply is realized, and the auctioneer (or, the seller) allocates the first infinitesimal unit to the bidder who submitted the highest bid, then the second infinitesimal unit to the bidder who submitted the second-highest bid, etc.[8] Each bidder pays her bid for each unit she obtains. The monotonic nature of how units are allocated implies that a collection of bids a bidder submitted can be equivalently described as a reported demand curve that is weakly-decreasing in quantity, but not necessarily continuous; the ultimate allocation resembles that of a classical Walrasian market, in which supply equals demand at a market-clearing price. We study pure-strategy Bayesian Nash equilibria of this auction.[9]

We establish a bound on the equilibrium market clearing price in terms of bidders' marginal values. The special cases of our bound are implicit in the equilibrium constructions in the parametric examples of pay-as-bid we discuss below, but ours is the first bound on all pure-strategy equilibria, the first bound which obtains in environments with asymmetric information, and the first bound that allows for mixed-strategy equilibria.[10] The bound

[7]See e.g. Hortaçsu and McAdams [2010] and Marszalec [2017], and our discussion below. An exception is the constrained strategic equilibrium approach developed by Armantier et al. [2008] and applied by Armantier and Sbaï [2006], among others.

[8]To fully-specify the auction we need to specify a tie-breaking rule; we adopt the standard tie-breaking rule, pro-rata on the margin, but our theory of equilibrium bidding does not hinge on this choice. This is in contrast to uniform-price auction, where tie-breaking matters; see Kremer and Nyborg [2004].

[9]In equilibrium, each bidder responds to the stochastic residual supply (that is, the supply given the bids of the remaining bidders). Effectively, the bidder is picking a point on each residual supply curve. In determining her best response, a bidder needs to keep in mind that: (i) the bid that is marginal if a particular residual supply curve is realized is paid not only when it is marginal, but also in any other state of nature that results in a larger allocation, and hence the bidder faces tradeoffs across these different states of nature; and (ii) bid curves need to be weakly monotonic in quantity, potentially a binding constraint.

[10]A different bound, in terms of competitive markets, was obtained by Swinkels [1999] for large economies. Our bound applies is valid in all finite markets.

plays a crucial role in our analysis of equilibria and in our asymmetric-information revenue comparisons.

We provide two sufficient conditions for equilibrium existence: a complex condition that is more general but difficult to analyze, and a simple condition that is less general but straightforward. Our simple condition reduces the existence question to checking optimization properties pointwise. It is satisfied, for instance, in the linear-Pareto settings analyzed by the prior literature discussed above, as well as for convex marginal values and for any distribution of supply provided there are sufficiently many bidders.[11] There is a large literature on equilibrium existence in pay-as-bid auctions. In symmetric-information settings, in addition to the contributions discussed above, Holmberg [2009] proves the existence of equilibrium when the distribution of supply has a decreasing hazard rate, and recognizes the possibility that (pure-strategy) equilibrium may not exist.[12] Our sufficient condition for existence encompasses the prior conditions and is substantially milder. In asymmetric information settings, Athey [2001], McAdams [2003], and Reny [2011] have shown that equilibrium exists in multi-unit (discrete) pay-as-bid auctions, and Woodward [2019a] established existence in the divisible-good context that we study.[13] A key difference between the results in these papers and ours is that the presence of private information allows the purification of mixed-strategy equilibria; such purification is not possible in the symmetric-information instances of our setting. Our existence conditions are consequences of our uniqueness and representation theorems, and (unlike general existence results) are not independent of the form of equilibrium.[14]

Our theorems establishing the existence and uniqueness of pure-strategy Bayesian Nash equilibrium in pay-as-bid auctions are reassuring for sellers using the pay-as-bid format; indeed, there are well-known problems posed by multiplicity of equilibria in other multi-unit auction formats.[15] Uniqueness is also important for the empirical study of pay-as-bid auctions. Estimation strategies based on the first-order conditions, or the Euler equation, rely on agents playing comparable equilibria across auctions in the data (Février et al. [2002],

[11]For many distributions of interest our condition is also satisfied with relatively few bidders; we provide examples in Section 3.

[12]See also Fabra et al. [2006], Genc [2009], and Anderson et al. [2013] for discussions of potential problems with equilibrium existence.

[13]For equilibrium existence in multi-unit auctions, see also Břeský [1999], Jackson et al. [2002], Reny and Zamir [2004], Jackson and Swinkels [2005], McAdams [2006], Břeský [2008], and Kastl [2012]. Milgrom and Weber [1985] show existence of mixed-strategy equilibria.

[14]In our design analysis, we show that transparency is optimal for the pay-as-bid auctioneer. Thus optimal pay-as-bid auctions always admit a unique equilibrium, which is in pure strategies.

[15]We establish the uniqueness of bids for relevant quantities—that is, for quantities a bidder wins with positive probability. Bids for quantities never obtained play no role in equilibrium outcomes. Our uniqueness result does not apply to these irrelevant bids.

Hortaçsu and McAdams [2010], Hortaçsu and Kastl [2012], and Cassola et al. [2013]).[16] Equilibrium uniqueness plays an even larger role in the study of counterfactuals (see Armantier and Sbaï [2006]).[17]

Uniqueness was studied by Wang and Zender [2002] who prove the uniqueness of "nice" equilibria under strong parametric assumptions on utilities and distributions. Assuming that marginal values are linear and that supply is drawn from an unbounded Pareto distribution, they analyzed symmetric equilibria in which bids are piecewise continuously-differentiable functions of quantities and supply is invertible from equilibrium prices; they showed the uniqueness of such equilibria. Holmberg [2009] restricted attention to symmetric equilibria in which bid functions are twice differentiable, and—assuming that the maximum supply strictly exceeds the maximum total quantity the bidders are willing to buy—proved the uniqueness of such smooth and symmetric equilibria.[18] Ewerhart et al. [2010] and Ausubel et al. [2014] independently expand these analyses to Pareto supply with bounded support and linear marginal values. Restricting attention to equilibria in which bids are linear functions of quantities, they showed the uniqueness of such linear equilibria. In contrast, we look at all Bayesian Nash equilibria of our model, we impose no parametric assumptions (not even continuity) and we do not require that some part of the supply is not wanted by any bidder.[19]

Our uniqueness result is also related to Klemperer and Meyer [1989] who established uniqueness in a duopoly model closely related to uniform-price auctions: when two symmetric and uninformed firms face random demand with unbounded support, then there is a unique equilibrium in their model.[20] The main difference between the two papers is, of course, that Klemperer and Meyer analyze the uniform-price format, while we look at pay-as-bid.[21]

[16] Maximum likelihood-based estimation strategies (e.g. Donald and Paarsch [1992]) also rely on agents playing comparable equilibria across auctions in the data. Chapman et al. [2005] discuss the requirement of comparability of data across auctions.

[17] See also, in a related context, Cantillon and Pesendorfer [2006].

[18] Holmberg's assumption that bidders do not want to buy part of the supply represents a physical constraint in the reverse pay-as-bid electricity auction he studies: in his paper bidders supply electricity and face capacity constraints—beyond a certain level they cannot produce more. This low-capacity assumption drives the analysis and it precludes directly applying the same model in the context of securities auctions in which bidders are always willing to buy more (provided the price is sufficiently low).

[19] As a consequence of this generality, we need to develop a methodological approach which differs from that of the prior literature. McAdams [2002] and Ausubel et al. [2014] have also established the uniqueness of equilibrium in their respective parametric examples with two bidders and two goods.

[20] The analogue of their unbounded support assumption is our assumption that the support of supply extends all the way to no supply. While the two assumptions look analogous they have very different practical implications. In a treasury auction, for example, a seller can guarantee that with some tiny probability the supply will be lower than the target; in fact, in practice the supply is often random and our support assumption is satisfied. On the other hand, it is substantially more difficult, and practically impossible, for the seller to guarantee the risk of arbitrarily-large supplies. Note also that we have known since Wilson [1979] that the uniform-price auction may admit multiple equilibria. No similar multiplicity constructions exist for pay-as-bid auctions.

[21] The proof of our uniqueness result follows a differential analysis familiar from uniqueness results for

Our bid representation theorem may be seen as a finite-market counterpart of Swinkels [2001], who studies pay-as-bid and uniform-price auctions in large markets, and in the limit, as the number of bidders goes to infinity, our representations are equivalent. He restricts attention equilibria that are asymptotically environmentally similar, an assumption we do not need. Our contribution also lies in establishing the representation of bids as averages of marginal values in all finite markets and not only in the limit. Holmberg [2009] derives a closed-form representation for symmetric and smooth equilibria subject to constraints on supply. We make no such assumptions, and instead prove that equilibria are symmetric and smooth; our results therefore provide support for his analysis and our finite-market representation of bids as weighted averages of marginal values is new.

Our bid representation result is surprising in the context of prior finite-market literature, which can be naturally read as suggesting that pay-as-bid equilibria are complex in the environments we study.[22] Prior constructions of finite-market equilibria focused on the setting in which bidders' marginal values are linear in quantity and the distribution of supply is (a special case of) the generalized Pareto distribution; see Wang and Zender [2002], Federico and Rahman [2003], Hästö and Holmberg [2006], Holmberg [2009], Ewerhart et al. [2010], and Ausubel et al. [2014]. This literature expressed equilibrium bids in terms of the intercept and slope of the linear demand and the parameters of the generalized Pareto distribution. Our treatment is not only more general but it also avoids the complexity inherent in expressing bids in terms of parameters of the functional forms studied in the earlier literature.

Our transparency result—that deterministic selling strategies are optimal—may appear familiar from the no-haggling theorem of Riley and Zeckhauser [1983]. However, in multi-object settings the reverse has been shown by Pycia [2006] and Manelli and Vincent [2006]; and, as mentioned above, nondeterministic supply may have a role in uniform-price auctions. Furthermore, there is a subtlety specific to pay-as-bid that might suggest a role for randomization: by randomizing supply below the monopoly quantity, the seller forces bidders to compete and bid more for these quantities, and in pay-as-bid the seller collects the raised bids even when the realized supply is near the monopoly quantity. We show that, despite these considerations, committing to deterministic supply is indeed optimal.[23]

first-price auctions (see, e.g., Maskin and Riley [2003], Lizzeri and Persico [2000], and Lebrun [2006]), but our analysis establishing the initial condition for the differential analysis is distinct.

[22]We focus our discussion on settings with decreasing marginal utilities; for constant marginal utilities see Back and Zender [1993], and Ausubel et al. [2014] among others.

[23]We further establish that the seller sets deterministic supply in all perfect Bayesian equilibria of the game in which the seller first designs a pay-as-bid auction and then the bidders bid; this is in contrast to uniform-price auction design games for which we construct equilibria in which the seller sets random supply (even though we show that the revenue-maximal equilibrium has deterministic supply in the uniform-price auction, as in the pay-as-bid auction). Note that Chen et al. [2019] show that individual outcomes of a given random mechanism can be replicated by a deterministic mechanism when there are multiple privately

We also establish a disclosure result: independent of the parameterization of the auction, the seller prefers to commit to announce the realization of supply prior to bid submission. Whether to reveal supply is an important question in treasury auctions, where the seller has pertinent information on supply prior to the auction.[24] The reason for this disclosure result (as well as the preceding transparency result) is a novel bound on revenues in pay-as-bid auctions with random supply, rather than Milgrom and Weber's [1982] celebrated linkage principle; the linkage principle is known to fail in the multi-unit auction context, cf. Perry and Reny [1999] and Vives [2010]. Furthermore, while our setting is one of Bayesian persuasion and information design, the full disclosure we establish stands in stark contrast to Kamenica and Gentzkow's [2011] paradigmatic insight that in information design and Bayesian persuasion the sender wants to withhold—or obfuscate—information. Related to information design, Bergemann et al. [2017] and Bergemann et al. [2019] also find the optimality of withholding information in single-unit auctions. The reason why there is no contradiction between these results and our finding the optimality of full disclosure is that their sender possesses information about bidders' valuations while in our analysis the bidders (receivers) are fully informed of their own value functions and the seller (sender) has and can release information about the quantity supplied, which is a key element of the bidders' strategic interaction.[25] Our disclosure results also contribute to the literature on dynamic mechanism design (cf., e.g., Pavan et al. [2014]).

While we are not aware of prior literature on optimal design of supply and reserve prices in pay-as-bid, the general mechanism design question was addressed by Maskin and Riley [1989]: what is the revenue-maximizing mechanism to sell divisible goods? The optimal

informed participants, while we show that not only can the maximal revenue generated by any random pay-as-bid auction be obtained by some deterministic mechanism, but also that this is possible without fundamentally changing the auction mechanism.

[24]The empirical impact of transparency has been extensively studied in the context of over-the-counter markets, for a recent review of this literature see e.g. Garratt et al. [2019]. The impact of transparency in uniform-price auctions has been experimentally studied by Hefti et al. [2019].

[25]In single-unit auctions bidders necessarily have full information regarding the quantity supplied, and the auctioneer's role in information design is inherently limited. Fang and Parreiras [2003] and Board [2009] study the limits of the linkage principle and the resulting benefits of information withdrawal or obfuscation. The optimality of obfuscation generally obtains in setting in which the participation constraints are interim and the seller cannot charge for information (cf. Bergemann and Pesendorfer [2007]). Even if the seller can charge for information, obfuscation is shown to be optimal by Li and Shi [2017] except under orthogonality assumptions of Eső and Szentes [2007]. Obfuscation is also established in other settings in which—like in our auction setting—the sender's interest (more revenue) is fundamentally misaligned with the bidders' interests (reducing payment); in a global games context see, e.g., Li et al. [2020]. For analysis of bidders' investment in information acquisition in auctions see e.g. Persico [2000] who finds that bidders in first-price auctions acquire more value-relevant information than bidders in second-price auctions. Finally, while we study a seller/sender who is able to commit to a disclosure strategy, our disclosure result immediately implies that a sender unable to commit would also fully reveal supply information. For information disclosure under no commitment see e.g. Grossman and Hart [1980] and Milgrom [1981].

mechanism they described is complex and in practice the choice seems to be between the much simpler auction mechanisms: pay-as-bid and uniform-price.[26] On the other hand, design issues have been addressed in the context of uniform-price auction. The design analysis of uniform-price focused on preventing collusive equilibria: Klemperer and Meyer [1989] point out that the auctioneer can induce competition in a uniform-price auction by introducing slight randomness in supply, Kremer and Nyborg [2004] look at the role of tie-breaking rules, LiCalzi and Pavan [2005] and Burkett and Woodward [2020b] at elastic supply, McAdams [2007] at commitment, and Burkett and Woodward [2020a] at the role of price selection. By proving equilibrium uniqueness for pay-as-bid we show its resilience to equilibrium collusion, thus providing a pay-as-bid counterpart for this literature. We also contribute to this uniform-price literature directly by showing that not only the seller but also the bidders might be made worse off by the possibility of tacit collusion; the reason is that the seller who expects a collusive equilibrium in uniform-price auction might optimally respond by setting a high reserve price, thus recovering some of the revenue at the cost of bidders' surplus.

Our revenue and welfare comparisons between pay-as-bid and uniform-price auctions contribute to the rich discussion of the pros and cons of these two formats. Swinkels [2001] focused on equilibria satisfying an asymptotic environmental similarity assumption and showed that pay-as-bid and uniform-price are revenue- and welfare-equivalent in large markets; Jackson and Kremer [2006] find revenue- and welfare- equivalence in large market limit under the assumption that the proportion of supply to the number of bidders vanishes to zero; our equivalence result does not rely on the size of the market, nor on an environmental similarity assumption, nor on extreme competition among bidders. Wang and Zender [2002] find pay-as-bid revenue superior in the equilibria of the complete-information linear-Pareto model their consider, and Woodward [2019b] extends this dominance to mixed-price combinations of pay-as-bid and uniform-price auctions. Ausubel et al. [2014] show that—with ex-ante asymmetric bidders with flat demands—either format can be revenue superior.[27] Our results on approximate revenue equivalence with small informational asymmetries complement this ambiguity: for uniform-price to raise significantly more revenue than pay-as-bid, bidders must be significantly asymmetric. In aggregate, prior theoretical work on the pay-as-bid versus uniform-price question has focused on revenue comparisons for fixed supply

[26] Furthermore, in the environments we focus on, the bidders' private information is correlated and hence the seller can nearly extract their full surplus using Crémer-McLean-type mechanisms [Crémer and McLean, 1988]; cf. footnote 58.

[27] When bidders have symmetric or non-flat demands, pay-as-bid is revenue superior in all examples they consider. The special supply distributions these papers consider are not revenue-maximizing, hence there is no conflict between their strict rankings and our revenue equivalence. See also Jackson and Kremer [2006] who find that—with non-optimized supply—either format can be revenue superior, and Anwar [1999] and Engelbrecht-Wiggans and Kahn [2002] for revenue comparisons with flat demands.

distributions and has allowed for neither reserve price nor supply optimization; indeed, the previous studies of pay-as-bid auctions with decreasing marginal values employed parametric specifications that did not support the analysis of design questions. Thus these results cannot address whether a well-designed pay-as-bid auction is preferable to a well-designed uniform-price auction. We go beyond these earlier papers both by allowing for the seller's optimization and by imposing no assumptions on the seller's information about the bidders.

Our divisible-good optimal revenue equivalence result provides a benchmark for the long-standing empirical debate whether pay-as-bid or uniform-price auctions raise higher expected revenues. This debate has attracted substantial empirical attention, with Hortaçsu and McAdams [2010] and Barbosa et al. [2020] finding no statistically significant differences in revenues, Février et al. [2002], Kang and Puller [2008], Armantier and Lafhel [2009], Marszalec [2017], and Mariño and Marszalec [2020] finding slightly higher revenues in pay-as-bid, and Castellanos and Oviedo [2008], Armantier and Sbaï [2006], and Armantier and Sbaï [2009] finding slightly higher revenues in uniform-price. Hortaçsu et al. [2018] argue that the revenues are similar.[28] As noted above, several of these papers conduct a counterfactual estimation of uniform-price revenues assuming truthful bidding, which is precisely the equilibrium selection under which our theoretical revenue and welfare equivalence obtains.[29]

Our results regarding the selection of auction format have other empirical implications. We show in our analysis of the auction design game that the auctioneer either strictly prefers a pay-as-bid auction or is indifferent between the pay-as-bid and uniform-price formats. All else equal, our model suggests that pay-as-bid auctions should be more prevalent than uniform-price auctions. This claim is supported by the multi-country analysis of security auction implementation in Brenner et al. [2009], which finds that pay-as-bid auctions are implemented by more than twice as many nations as implement uniform-price auctions, as well as the analysis of electricity markets in Del Río [2017], which finds that pay-as-bid auctions represent nearly 90% of electricity auctions. Additionally, counterfactual analysis of uniform-price auctions assumes truthful reporting of values to obtain an upper bound

[28] They note that bids in U.S. Treasury auctions are typically "flat" and infer that not much surplus is retained by bidders; an alternative explanation highlighted by our analysis is that the bidders are close to being completely informed. Note also that while flatness implies that there is not much difference between the revenues generated by the pay-as-bid and uniform-price auctions, the uniform-price auction brings large downside potential in the form of collusive-seeming behavior. Our uniqueness theorem shows that the pay-as-bid format mitigates this risk. Of note in this context is Häfner [2020], who empirically demonstrates that bidders overbid in pay-as-bid auctions.

[29] We show that revenue is maximized in the uniform price auction when the seller offers a deterministic quantity for sale, and bidders bid their true marginal values for the quantity they obtain. Whether they bid their true marginal values for quantities they do not obtain is irrelevant in equilibrium, thus we refer to truthful bidding at the received quantity as truthful bidding. This assumption is satisfied in some counterfactual approaches, where a bid curve is truthful if it matches the bidder's marginal value curve.

on unobserved revenue. Our results can be taken to show that this bound is tight only if bidders are playing a seller-optimal equilibrium; otherwise there may be a significant divergence between observed revenue and counterfactual predictions. Additionally, since revenue-dominance of the pay-as-bid auction implies that a seller should implement the uniform-price format only if she expects this favorable equilibrium to be played, we should expect counterfactual analysis from witnessed uniform-price auctions to find approximate revenue equivalence.

In our supplementary note [Pycia and Woodward, 2020] we provide additional results and applications that complement the present paper. Complementing the equilibrium uniqueness for pay-as-bid, in the supplementary note we analyze equilibrium multiplicity in uniform-price auctions. We also apply our transparency result to study the relationship of the pay-as-bid auctioneer to a classical monopolist. We show that the auctioneer's design problem is separable, and that the decisions of optimal supply and optimal reserve may be analyzed independently and show that this implies that increased variance of bidder values is good for the seller.

Finally, let us note that our analysis of pay-as-bid auctions can be reinterpreted as a model of dynamic oligopolistic competition among sellers who at each moment of time compete à la Bertrand for sales and who are uncertain how many more buyers are yet to arrive. Prior sales determine the production costs for subsequent sales, thus the sellers need to balance current profits with the change in production costs in the future. This methodological link between pay-as-bid auctions and dynamic oligopolistic competition is new, and we develop it in follow up work.[30]

2 Model and a Market Price Bound

There are $n \geq 2$ bidders, $i \in \{1, ..., n\}$. Bidder i's marginal valuation for quantity q is denoted $v^i(q; s_i)$, where s_i is a signal privately known to bidder i. Without loss of generality, we decompose the signals as $s_i = (s, \theta_i)$, where s is common to all bidders and θ_i is private to bidder i. We assume that each $v^i(\cdot; s_i)$ is strictly decreasing where it is strictly positive, Lipschitz continuous, and almost-everywhere differentiable in q, and, with the exception of Theorem 1, we study the symmetric case $v^i(q; s_i) = v(q; s_i)$. We allow arbitrary dimen-

[30]The oligopolistic sellers uncertain of future demands correspond to bidders in the pay-as-bid auction, and sellers' costs correspond to bidders' values. For prior studies of dynamic competition see e.g. Deneckere and Peck [2012]; while they study competition among a continuum of sellers, the pay-as-bid-based approach allows for the strategic interaction between a finite number of sellers. The other canonical multi-unit auction format, the uniform-price auction, was earlier interpreted in terms of static oligopolistic competition by Klemperer and Meyer [1989].

sionality of s_i, and an arbitrary integrable $v(q;\cdot)$. The seller is uninformed and we study environments in which s_i are highly correlated across bidders: in Sections 3–5 we analyze the case when the correlation is perfect, $s_1 = ... = s_n = (s, 0)$, without imposing any further assumptions on the distribution of s; in Section 6 we relax the perfect correlation assumption and show that the insights of Sections 3–5 are robust to this relaxation. Under perfect correlation, signal s has no strategic importance for bidders participating in an auction, and thus when studying the equilibrium among such bidders in Section 3, we fix s and denote the bidders' marginal valuation by $v^i(q; s_i) = v(q)$. Bidders' information plays an important role in the analysis of the seller's problem in Sections 4, 5, and 6.[31]

To simplify the exposition of the design problem, we normalize the seller's cost to 0. Our insights do not hinge on this normalization, and remain valid for any convex increasing cost function.[32] Our design analysis builds on the existence, uniqueness, and bid representation results for pure-strategy Bayesian Nash equilibria of the pay-as-bid auction. We thus start by analyzing such equilibria. In the equilibrium analysis we study supply Q drawn from distribution F with density $f > 0$ and support $[0, \overline{Q}]$; we also allow F with full mass concentrated at one point. Q is independent of the bidders' signal s.[33] Otherwise we impose no global assumptions on F. In our analysis of auction design, the seller is free to choose any such distribution F as long as $\overline{Q} \leq Q^{\max}$, where $Q^{\max}$ is the maximum supply available to the seller.[34]

The seller implements a reserve price R. We denote by $\overline{Q}^R = Q^R(\overline{Q}, s, \theta)$ the effective maximum quantity allocated, equal to the $\overline{Q}$ if the reserve is not binding and equal to the amount demanded at price R by bidders with signals s and $\theta = (\theta_1, ..., \theta_n)$ when the reserve is binding. More formally, and allowing any $Q \leq \overline{Q}$, we define $Q^R\left(Q, s, \theta\right) = Q$ if $R = 0$, and $Q^R\left(Q, s, \theta\right) = \min\left\{Q, \sum_{i=1,..,n} v^{-1}\left(R; s, \theta_i\right)\right\}$ if $R > 0$ (note that the inverse v^{-1} is well defined for $R > 0$).

In the pay-as-bid auction, each bidder submits a weakly decreasing bid function $b^i(q)$: $[0, \overline{Q}] \to \mathbb{R}_+$. Without loss of generality we may assume that the bid functions are right-

[31] The seller may not know the bidders' information if, for example, the seller needs to commit to the auction mechanism before this information is revealed. Alternatively, the seller may want to fix a single design for multiple auctions.

[32] The reason why more general cost functions do not substantively change the analysis is that the Transparency Theorem (Theorem 6), on which the analysis of design builds, is valid irrespective of seller's cost function. We provide more detailed discussion in Section 4.

[33] This last assumption is not needed in our analysis of elastic supply (see Appendix A). Q might be an on-path or off-path supply in seller's design problem or it might represent e.g. supply net of non-competitive bids as discussed in Back and Zender [1993], Wang and Zender [2002], and subsequent literature.

[34] We could allow for infinite $Q^{\max}$ as long as the optimal monopoly quantity remains finite. This would be so if, e.g., the seller faces increasing and convex marginal costs of supply, or there is no heavy tail of marginal values.

continuous.[35] The auctioneer then sets the market price $p^\star$ (also known as the stop-out price),

$$p^\star = \max\left\{R, \sup\left\{p'\colon q_1 + ... + q_n \geq Q \text{ for all } q_1, \ldots, q_n \text{ such that } \; b^1\left(q_1\right), ..., b^n\left(q_n\right) \leq p'\right\}\right\}.$$

If the set over which the supremum is taken is empty, then the stop-out price is set to the reserve price R. Agents are awarded a quantity associated with their demand at the stop-out price,

$$q_i = \max\left\{q'\colon b^i\left(q'\right) \geq p^\star\right\},$$

as long as there is no need to ration them. When necessary, we ration pro-rata on the margin, the standard tie-breaking rule in divisible-good auctions. The details of the rationing rule have no impact on the analysis of equilibrium bidding we pursue in Section 3.[36] The demand function (the mapping from p to q^i) is denoted by $\varphi^i(\cdot)$.[37] Agents pay their bid for each unit received, and utility is quasilinear in monetary transfers; hence,

$$u^i\left(b^i\right) = \int_0^{q^i(p^\star)} v\left(x\right) - b^i\left(x\right) dx.$$

2.1 A Bound on Market Price

Our analysis of optimal bidding relies on the following key theorem proven in Supplementary Appendix B; in this theorem we impose no restrictions on bidders' information and we allow mixed-strategy equilibria.

Theorem 1. [A Bound on Market Price] *In any mixed-strategy equilibrium of the pay-as-bid auction, for any signal profile* $(s, (\theta_1, \ldots, \theta_n))$ *all realizations of the market clearing price for the effective maximum quantity* $\overline{Q}^R$ *are bounded between the smallest and largest*

[35]This assumption is without loss because we study a perfectly-divisible good and we ration quantities pro-rata on the margin. Indeed, we could alternatively consider an equilibrium in strategies that are not necessarily right-continuous. By assumption, the equilibrium bid function of a bidder is weakly decreasing, hence by changing it on measure zero of quantities we can assure the bid function is right continuous. Such a change has no impact on this bidder's profit, or on the profits of any of the other bidders, because rationing pro-rata on the margin is monotonic in the sense of footnote 36. In fact, there is no impact on bidders' profits even conditional on any realization of Q.

[36]The only place when we rely on rationing rule is the analysis of reserve prices but even in this part of the analysis all we need is that rationing rule is monotonic: that is, the quantity assigned to each bidder increases when the stop-out price decreases; rationing pro-rata on the margin satisfies this property.

[37]Where $b^i(\cdot)$ is constant, φ^i is not well-defined. Where important, we will use $\underline{\varphi}^i$ and $\overline{\varphi}^i$ to denote the right- and left-continuous (respectively) inverses of b, $\underline{\varphi}^i\left(p\right) = \sup\left\{q\colon b^i\left(q\right) > p\right\}$ and $\overline{\varphi}^i\left(p\right) = \sup\left\{q\colon b^i\left(q\right) \geq p\right\}$.

marginal value at the per-capita effective maximum quantity,

$$\min_i \operatorname*{ess\,inf}_{\tilde{\theta}_i} v^i \left(\frac{1}{n} \overline{Q}^R ; s, \tilde{\theta}_i \right) \le p \left(\overline{Q}^R ; s, \theta \right) \le \max_i \operatorname*{ess\,sup}_{\tilde{\theta}_i} v^i \left(\frac{1}{n} \overline{Q}^R ; s, \tilde{\theta}_i \right).$$

The proof of Theorem 1 shows a slightly stronger claim: for any realization of (s, θ), the equilibrium bid for the maximum quantity bidder i, with type $s_i = (s, \theta_i)$, can obtain equals the bidder's marginal value for this quantity; or, bids equal values at the maximum feasible quantity. The intuition for this claim is that if a bidder has strictly positive margin at the maximum feasible quantity, they can slightly increase their bid and obtain a non-negligible additional quantity at minimally higher price, which is a profitable deviation; the proof of Theorem 1 formalizes this intuition and takes care of technical complications related to tie-breaking, flat bids, and binding monotonicity constraints. Note that this intuition applies only to the maximum quantity at which the increased bid is paid only when it is marginal; at any lower quantity the increased bid would need to be paid also when inframarginal.

Because bids are decreasing in quantity, the equilibrium market-clearing price is minimized when realized quantity is maximized, $Q = \overline{Q}^R$. Thus, the theorem provides bounds on the minimum market price. Furthermore, when bidders are symmetric and have only common and no idiosyncratic information, then

$$\min_i \operatorname*{ess\,inf}_{\tilde{\theta}_i} v^i \left(\frac{1}{n} \overline{Q}^R ; s, \tilde{\theta}_i \right) = v \left(\frac{1}{n} \overline{Q}^R ; s \right) = \max_i \operatorname*{ess\,sup}_{\tilde{\theta}_i} v^i \left(\frac{1}{n} \overline{Q}^R ; s, \tilde{\theta}_i \right)$$

That is, the market price at the maximum quantity is exactly equal to each bidder's marginal value at the last unit they receive; this equality is illustrated in Figure 1 in Section 3 below.

Because bids equal values at bidders' maximum feasible quantities, and these quantities are sold at the maximum realized supply, the equality of market price and bidders' marginal values obtains at the maximum realized supply. The market-clearing price is minimized at this supply, but the lower bound of Theorem 1 remains valid irrespective of the realization of supply. The market-clearing price at supply lower than $\overline{Q}^R$ can (and frequently does) rise above $\max_i \operatorname{ess\,sup}_{\tilde{\theta}_i} v^i(\overline{Q}^R/n; s, \tilde{\theta}_i)$.

Theorem 1 plays a crucial role in our equilibrium uniqueness result for symmetrically informed bidders, and hence in many of our subsequent results. Theorem 1 also plays a key role in our bounds on pay-as-bid revenues beyond the symmetrically-informed bidders case as well as in our revenue comparison of pay-as-bid and uniform-price auctions.

3 Pay-as-Bid Equilibrium

We start our analysis by establishing novel and general results for the benchmark case when bidders are symmetrically informed, $\theta_1 = ... = \theta_n$. We relax this assumption in Section 6. When analyzing an equilibrium of the pay-as-bid auction, signals s and θ have no strategic importance for bidders and thus we hold them fixed, and denote the bidders' symmetric marginal valuation by $v^i(q; s_i) = v(q)$.

3.1 Existence, Uniqueness, and Bid Representation

We first show that equilibrium is unique and tractable. The existence of equilibrium can then be analyzed in terms of what equilibrium strategies must be, if an equilibrium exists. We therefore defer discussion of existence until after our uniqueness and representation results, and for expositional simplicity our uniqueness and representation results are formulated conditional on the existence of Bayesian Nash equilibrium. Furthermore, these results constrain attention to relevant quantities at which bids can possibly affect utility; bids for quantities which the bidder never receives must be weakly decreasing and sufficiently competitive, but are not typically uniquely determined.[38] Proofs of all results may be found in Supplementary Appendix C.

Theorem 2. **[Uniqueness]** *The Bayesian Nash equilibrium is unique.*

For an intuitive approach to this theorem, notice that if we restricted attention to symmetric and smooth equilibria (which we do not), then uniqueness would follow from Theorem 1. Indeed, in a symmetric smooth equilibrium bidders' first-order conditions give us an ordinary differential equation and Theorem 1 provides us with a unique initial condition for this equation by uniquely determining the price $p(\overline{Q}^R)$ at the maximum supply and hence, in a symmetric equilibrium, the bids for quantity $\overline{Q}^R/n$. The proof, provided in Supplementary Appendix C, builds on this idea and addresses the difficulties raised by potential asymmetries, non-differentiabilities, and discontinuities.

Equilibrium uniqueness leads us to the bid representation theorem, expressed in terms of *weighting distributions*. For any quantity $Q \in [0, \overline{Q})$, the n-bidder weighting distribution of F has c.d.f. $F^{Q,n}$ that increases from 0 when $x = Q$ to 1 when $x = \overline{Q}$. This c.d.f. is given

[38]The reason a bidder's bids on never-won quantities need to be sufficiently competitive is to ensure that other bidders do not want decrease their bids on relevant quantities. With a binding reserve price, the bids on never-won quantities may not need to be competitive and hence these bids are even less determined, but the equilibrium bids on the relevant quantities—those which are sometimes marginal—remain uniquely determined. Importantly, these bids being insufficiently competitive does not induce alternate equilibria: there are no equilibria in which these bids are lower than required to support the unique equilibrium we find.

by

$$F^{Q,n}(x) = 1 - \left(\frac{1-F(x)}{1-F(Q)}\right)^{\frac{n-1}{n}}.$$

The auxiliary c.d.f.s $F^{Q,n}$ play a central role in our bid representation theorem below. These distributions depend only the number of bidders and the distribution of supply, and not on any bidder's true demand. As the number of bidders increases the weighting distributions put more weight on lower quantities.

Theorem 3. **[Bid Representation]** *The unique equilibrium is symmetric. For any quantity $q \in [0, \overline{Q}^R/n]$, the bid b^i of each bidder i is given by*

$$b^i(q) = \int_{nq}^{\overline{Q}} v\left(\frac{\min\left\{x, \overline{Q}^R\right\}}{n}\right) dF^{nq,n}(x). \tag{1}$$

For any quantity $Q \in [0, \overline{Q}]$ the resulting market-clearing price function is given by

$$p(Q) = \int_{Q}^{\overline{Q}} v\left(\frac{\min\left\{x, \overline{Q}^R\right\}}{n}\right) dF^{Q,n}(x). \tag{2}$$

When the reserve price does not bind, formulas (1) and (2) simplify, as $\overline{Q}^R = \overline{Q}$ and $\min\left\{x, \overline{Q}^R\right\} = x$.[39]

Recall that we impose no assumptions on symmetry of equilibrium bids, their strict monotonicity, nor continuity or differentiability; we derive all these properties. Furthermore, the equilibrium bids b^i are appropriately-weighted averages of bidders' marginal values v, and in this they resemble the bids in first-price auctions with privately-informed bidders. Because marginal values are decreasing in quantity, bids are below values—that is, bidders are shading their bids—except for the bid on the effective maximum quantity where limit equality obtains, an equality consistent with Theorem 1. Because the unique equilibrium is symmetric, the market price $p(Q)$ given supply Q and the bid functions b^i are related in a natural way, $b^i(q) = p(nq)$.

Consider three examples. Substitution into our bid representation shows that when marginal values v are linear and the supply distribution F is generalized Pareto, $F(x) =$

[39] Because $\min\{x, \overline{Q}^R\}$ is constant for $x > \overline{Q}^R$, the equilibrium bid equation can be rewritten as

$$b^i(q) = \int_{\min\left\{nq, \overline{Q}^R\right\}}^{\overline{Q}^R} v\left(\frac{x}{n}\right) dF^{nq,n}(x) + v\left(\frac{\overline{Q}^R}{n}\right)\left(1 - F^{nq,n}\left(\overline{Q}^R\right)\right).$$

The equilibrium market price equation can be expressed similarly.

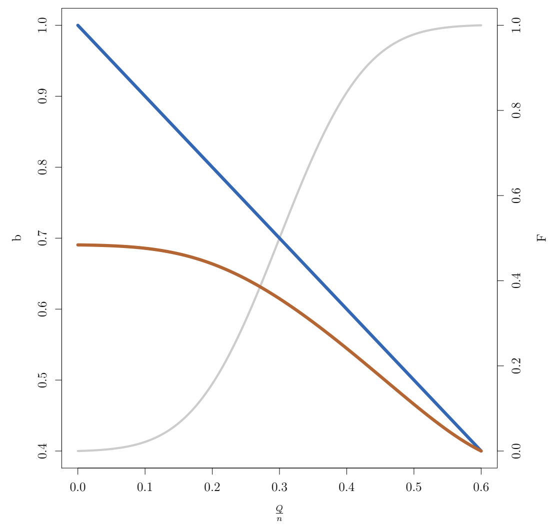

Figure 1: Equilibrium bids when values are linear and the distribution of supply Q is truncated normal.

$1-\left(1-\frac{x}{\overline{Q}}\right)^{\alpha}$ for some $\alpha > 0$, the equilibrium bids are linear in quantity. This case of our general setting has been analyzed by Ewerhart et al. [2010], and Ausubel et al. [2014].[40] Our bid representation remains valid when F has a mass point at $\overline{Q}$, and this insight is crucial to our analysis of reserve prices. This is true even when the distribution is degenerate and puts all its mass on $\overline{Q}$: taking the limit of continuous probability distributions which place increasingly more probability near $\overline{Q}$, the representation implies that equilibrium bids are flat, as they should be (see Corollary 1). Finally, Figure 1 illustrates the equilibrium bids for ten bidders with linear marginal values who face a distribution of supply that is truncated normal. This and the subsequent figures represent bids, marginal values, and the c.d.f. of supply; it is easy to distinguish between the three curves since bids and the marginal values are decreasing (and bids are below marginal values) while the c.d.f. is increasing.[41]

Theorem 4. [Existence] *There exists a pure-strategy Bayesian Nash equilibrium whenever for any* $q \in [0, \frac{1}{n}\overline{Q}^R]$, $(v(q)-b)(1-F(q+(n-1)\varphi(b))$ *is single-peaked on* $b \in [p(\overline{Q}^R), v(q)]$.

The algebraic expression in Theorem 4 is the bidder's pointwise first-order condition for optimal bidding (this expression is derived in Appendix C). Then Theorem 4 states that equilibrium exists if at each relevant quantity q the bid $b^i(q)$ satisfies the standard

[40] They calculate bid functions in terms of the parameters of their model (linear marginal values and Pareto distribution of supply) and do not rely on or recognize the representation of bids as weighted averages that is crucial to our subsequent analysis.

[41] In all figures, we check our equilibrium existence condition and calculate bids numerically using R. In Figure 1 we use a normal distribution with mean 3 and standard deviation 1, truncated to the interval $[0, 6]$.

second order condition of bidder i's quantity-by-quantity (pointwise) utility optimization. The result follows from familiar arguments in single-dimensional contexts. Ignoring the constraint that bids must be weakly decreasing in quantity, if a bid function solves the bidder's optimization pointwise then it is a global maximum and a best response. Given a quantity q, if the pointwise objective is single-peaked on the range of feasible prices there is at most one bid at which the best response first-order condition is satisfied. Since bids in the symmetric equilibrium given in Theorem 3 solve these first-order conditions, they are best responses. Note that Theorem 3 implies that the inverse bid φ is expressible in terms of model fundamentals, so Theorem 4 is not conditioning equilibrium existence on an equilibrium object. We provide a proof and additional comments on our existence condition in Supplementary Appendix C.3.

Consider some examples. Our sufficient condition is satisfied when marginal values v are linear and F is a uniform distribution or a generalized Pareto distribution, $F(x) = 1-\left(1-\frac{x}{Q}\right)^{\alpha}$ where $\alpha > 0$.[42] When marginal values have slope bounded away from zero, this condition is also satisfied for any twice-differentiable c.d.f. F provided there are sufficiently many bidders.[43] And, the sufficient condition is satisfied whenever the inverse hazard rate H is increasing—hence when the hazard rate is decreasing—irrespective of the marginal value function v.[44] This follows since the left- and right-hand terms of the objective are decreasing in b (φ is decreasing in b). In the sequel we illustrate our other results with additional examples in which a pure-strategy equilibrium exists.[45]

While our sufficient condition shows that equilibrium exists in many cases of interest, there are situations in which the equilibrium does not exist; see our discussion in the introduction.[46]

[42]The existence of equilibrium in the linear/generalized Pareto example was independently established by Ewerhart et al. [2010] and Ausubel et al. [2014] for bounded generalized Pareto distributions and Wang and Zender [2002], Federico and Rahman [2003], and Holmberg [2009] for unbounded Pareto distributions.

[43]This is easiest to see in the weaker but more technical condition preceding the proof of Theorem 4 in Supplementary Appendix C.3. Holding F constant, if there is a (Q, p) at which the implication is violated then if there are $n' < n$ bidders there is a (Q', p') such that the condition is also violated, since $p(\cdot)$ is increasing in n. Holding per-capita supply constant, $Y_q \to 0$ for almost every Q, so when $v_q > \varepsilon > 0$ the implication will be satisfied. In the limit where per-capita supply goes to zero, equilibrium existence was recognized by Swinkels [1999] and Jackson and Kremer [2006].

[44]The sufficiency of decreasing hazard rate for equilibrium existence was established by Holmberg [2009].

[45]The first of these examples features a normal distribution, the second one features strictly concave marginal values, and the last one features reserve prices. In general, our existence condition is closed with respect to several changes of the environment: adding a bidder preserves existence, making the marginal values less concave (or more convex) preserves existence, and raising the reserve price preserves existence.

[46]The construction of a tighter existence condition is complicated by the possibility of monotonicity-constrained deviations from the symmetric solution to the market clearing equation provided in Theorem 3. A global best response might exist which is the aggregation of nonoptimal local behavior. Our sufficient condition implies that the optimization problem is single-peaked in bid, and there is a unique global optimum.

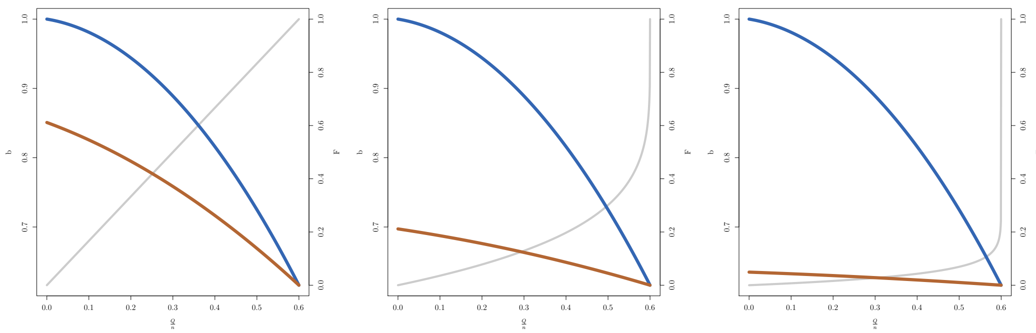

Figure 2: Bids are flatter for more concentrated distributions of supply.

The rest of our paper builds upon the above results to establish qualitative properties of the unique equilibrium, and to provide guidance as to how to design divisible good auctions including for environments in which bidders' information is not perfectly correlated.

3.2 Equilibrium Properties and Comparative Statics

The bid representation of Theorem 3 has many natural implications. We discuss them in this subsection and provide a related link between revenue and the distribution of private information in Section 6.

A case of particular interest arises when the distribution of supply is concentrated near some target quantity. We say that a distribution is δ-concentrated near quantity Q^* if $1-\delta$ of the mass of supply is within δ of quantity Q^*. Our bid representation theorem implies that the bids on initial quantities are nearly flat for concentrated distributions.

Corollary 1. **[Flat Bids]** *For any $\varepsilon > 0$ and quantity Q^* there exists $\delta > 0$ such that, if supply is δ-concentrated near $Q^* \leq \overline{Q}^R$, then the equilibrium bids for all quantities lower than $\frac{Q^*}{n} - \varepsilon$ are within ε of $v\left(\frac{Q^*}{n}\right)$.*

Figure 2 depicts the flattening of equilibrium bids predicted by Corollary 1; in the three sub-figures ten bidders face supply distributions that are increasingly concentrated around the total supply of 6 (per capita supply of 0.6).

Remark 1. To illustrate the value of Corollary 1, let us consider the analysis of U.S. Treasury auctions for short-term securities in Hortaçsu et al. [2018]. In these auctions supply randomness is low, and empirically-observed uniform-price bids are nearly flat. Because supply randomness is low, Corollary 1 implies that counterfactual pay-as-bid bids would also be nearly flat, and changing the auction format would yield little additional revenue.[47]

[47] Hortaçsu et al. [2018] take a different approach, and use inferred marginal values to show that bidders

Our representation theorem has also implications for bidders' margins. In the corollary below we refer to the supremum of quantities the bidder wins with positive probability as the highest quantity a bidder can win in equilibrium.

Corollary 2. **[Low Margins]** *The highest quantity a bidder can win in equilibrium is* $\frac{1}{n}\overline{Q}^R$*, and the bid at this quantity equals the marginal value,* $b\left(\frac{1}{n}\overline{Q}^R\right) = v\left(\frac{1}{n}\overline{Q}^R\right)$*. Furthermore, for any* $\varepsilon > 0$ *and quantity* $Q^* \leq \overline{Q}^R$ *there exists* $\delta > 0$ *such that, if supply is* δ*-concentrated near* Q^**, then each bidder's equilibrium margin* $v\left(\frac{1}{n}Q^* - \delta\right) - b\left(\frac{1}{n}Q^* - \delta\right)$ *on the* $\frac{1}{n}Q^* - \delta$ *unit is lower than* ε*.*

Thus, each bidder's margin on the last unit they could win is zero; this result follows from our bid representation theorem, as well as Theorem 1. Additionally, if supply is concentrated around some quantity Q^*, then the margin on units just below $\frac{1}{n}Q^*$ is close to zero.

Finally, bidders' equilibrium margins are lower and the seller's revenue is larger when there are more bidders:

Corollary 3. **[More Bidders]** *Bidders submit higher bids and the seller's revenue is larger and each bidder's profits smaller when there are more bidders—both when the supply distribution is held constant, and when the per-capita supply distribution is held constant.*

The corollary follows because as the number of bidders increases, $1-F^{Q,n}(x) = \left(\frac{1-F(x)}{1-F(Q)}\right)^{\frac{n-1}{n}}$ decreases, and hence $F^{Q,n}(x)$ increases, thus mass in the weighting distribution is shifted towards lower x, where marginal values are higher. At the same time, the marginal value at x either increases in n (if we keep the distribution of supply constant) or stays constant (if we keep the distribution of per-capita supply constant).

While bidders raise their bids when facing more bidders even if the per-capita distribution stays constant, our bid representation theorem implies that the changes are small.[48] This is illustrated in Figure 3 in which increasing the number of bidders from 5 bidders to 10 bidders has only a small impact on the bids, as does the further increase from 10 bidders to 5 *million* bidders.

do not obtain much surplus; thus changing in the auction format cannot yield much additional revenue. Our result is sharper: given flat uniform-price bids and relatively certain supply, changing the auction format also cannot cost much revenue.

[48]In the limit, and absent a reserve price, bids take the simple form $b(q) = \int_q^{\max \mathrm{Supp}\, \overline{F}} v(x)\, d\overline{F}(x) / \left(1 - \overline{F}(q)\right)$ where $\overline{F}$ is distribution of per-capita supply. Notice that if we keep the supply distribution fixed while more and more bidders participate in the auction, then in the large market limit revenue converges to average supply times the value on the initial unit. See Swinkels [2001] for limit results with fixed per-capita supply and Jackson and Kremer [2006] for limit results with fixed supply.

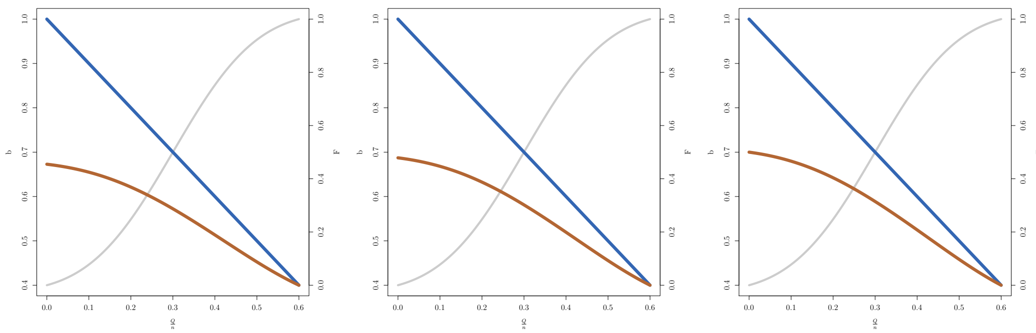

Figure 3: Bids go up when more bidders arrive (and per capita quantity is kept constant) but not by much: 5 bidders on the left, 10 bidders in the middle, and 5 million bidders on the right. Note that all axis scales are identical.

4 Designing Pay-as-Bid Auctions: Transparency and Disclosure

In this section we maintain the assumption that the pay-as-bid format is run and analyze the design of such auctions. We focus on the reserve price and the distribution of supply, the two natural elements of pay-as-bid auction that the seller can select.[49] For now, we consider the case in which the bidders observe the same signal s, while the seller does not know the bidders' signal and has only a belief about it, $s \sim \sigma$. In Section 6 we show that our insights are robust to the introduction of small informational asymmetries among bidders. As design decisions are taken from the seller's perspective, our terminology in this and the subsequent sections now explicitly keeps track of the bidders' information. We impose no assumptions on the distribution σ other than $v(q; \cdot)$ being integrable.

4.1 Transparency

The key insight that underlies our design analysis is that—in contrast to typical multidimensional mechanism design problems discussed in the introduction—in an optimized pay-as-bid auction deterministic—and, hence, transparent—supply is optimal. Furthermore, if supply is exogenously random, then it is optimal for the seller set a deterministic supply cap; and, independent of whether a supply cap is feasible, it is optimal to announce the realized supply to the bidders prior to the auction.

First, suppose that the seller has some deterministic quantity $\overline{Q}$ of the good; we relax

[49]In Appendix A, we show that our design insights extend to the environment in which the seller is not restricted to setting a reserve price and supply distribution and instead the seller can set a joint distribution over both. We simultaneously extend our analysis to allow for mixed-strategy equilibria.

this assumption below. For any fixed reserve price, we consider the problem of designing a supply distribution F that maximizes the seller's revenue. The seller has the option to offer a stochastic distribution over multiple quantities, and it is plausible that such randomization could increase his expected revenue. For instance, offering quantities lower than the optimal monopoly quantity, $Q^\star$, results in a tradeoff: the seller sometimes sells less than $Q^\star$, with a direct and negative revenue impact, but when he sells quantity above $Q^\star$ he will receive higher payments due to the pay-as-bid nature of the auction. This tradeoff is illustrated in Figure 2, in which concentrating supply lowers the bids.[50]

We show that selling the deterministic supply $Q^\star$ is in fact revenue-maximizing across all pure-strategy equilibria; for this reason in the sequel we refer to $Q^\star$ as optimal supply. In this section, we restrict attention to pure-strategy equilibria and, relatedly, maintain our global restrictions on the support of supply. In Appendix A, we relax these and other restrictions—e.g., allowing elastic supply—and prove that the pure-strategy equilibrium under transparent and deterministic supply revenue dominates any mixed-strategy equilibrium at any random supply.[51]

Theorem 5. [Transparency of Optimal Supply] *In pure-strategy equilibria, the seller's revenue under non-deterministic supply is strictly lower than under optimal deterministic supply. Optimal deterministic supply is given by the solution to the monopolist's problem when facing uncertain demand.*

As the following proof sketch indicates, Theorem 5 remains valid if the reserve price is arbitrary rather than optimized. The transparency result also remains valid for sellers who maximize profits equal to revenue net of costs, provided the marginal cost curve is weakly increasing.[52] Such sellers optimally choose the deterministic quantity (or quantity cap) that maximizes the expected revenue minus cost rather than the quantity that maximizes the expected revenue. Taking the cost into account affects what quantity is optimal, but it does not change the result that optimal supply is deterministic.

[50] A priori such trade-offs can go either way; see Pycia [2006] and the introduction.

[51] The restriction to pure-strategy equilibria can be also straightforwardly relaxed in the special case in which bidders have no private information. Furthermore, as pay-as-bid is largely employed by central banks and governments, the efficiency of allocations may be an important concern and a reason a seller may want to ensure that a pure-strategy equilibrium is being played. The symmetry of equilibrium strategies we prove in Theorem 3 implies that in pure-strategy equilibrium the marginal value for any unit received is higher than the marginal value for any unit not received. In a pure-strategy equilibrium, there are thus no efficiency improving re-allocations of units among bidders; this property trivially fails in any mixed-strategy equilibrium that is not essentially in pure strategies.

[52] In the absence of the increasing marginal cost assumption, an analogue of Theorem 5 would need to be modified to take account of resulting ironing. See also our remark at the end of the proof of the theorem in Supplementary Appendix E.

Remark 2. Equilibrium multiplicity in uniform-price auctions implies that the the optimality of transparent supply in pay-as-bid auctions does not extend to uniform-price auctions. The reason is that the bidding equilibrium may have an irregular dependence on the reserve price and supply distribution in the uniform-price auction. We discuss the issue in the ensuing analysis of the auction design game; cf. our discussion of Lemma 2.

To prove Theorem 5, we start with an arbitrary reserve price and supply distribution and the induced pure-strategy equilibrium bids. Holding equilibrium bids fixed, we use our bid representation from Theorem 3 to bound expected revenue by the standard monopoly revenue given the supply distribution.[53] In effect we obtain the following bound on the expected revenue,

$$\mathbb{E}_{s,Q}\left[\pi^{F}\left(Q;s\right)\right] \leq \int_{0}^{\overline{Q}^{R}} \mathbb{E}_{s}\left[\pi^{\delta}\left(Q;s\right)\right] dF\left(Q\right), \tag{3}$$

where $\pi^{F}(Q;s)$ is the seller's revenue when bidders' signal is s, the realization of supply is Q, and bidders bid against the distribution of supply F, while $\pi^{\delta}(Q;s)$ is the seller's revenue when bidders' signal is s, the realization of supply is Q, and bidders bid against the distribution of supply that puts probability 1 on supply quantity Q.[54] Note that $\pi^{\delta}(Q;s)$ is a monopolist's profit from selling quantity Q to buyers with common signal s. This upper bound implies that the seller's revenue is maximized when the seller sets the supply to be always equal to the revenue-maximizing deterministic supply. We provide the details of the proof in Supplementary Appendix E (bound (3) above restates inequality (10) in the proof).

The structure of the proof of Theorem 5 has two important implications. First, under the additional restriction that $Q\mathbb{E}_s[v^{-1}(Q;s)]$ is single-peaked in Q, the proof is applicable to environments in which the seller's underlying supply is random and the seller can lower the supply but cannot increase it above the underlying supply realization. In this more general environment we assume that the distribution of underlying supply is exogenously given by F with a compact support.[55] Our proof then shows that the revenue maximizing-supply reduction by the seller reduces supply to $Q^{\star}$ whenever the exogenous supply is higher than $Q^{\star}$, and otherwise leaves the supply unchanged.

[53] This argument hinges on re-assigning the revenue across supply realizations; in particular, the actual revenue conditional on a supply realization is not necessarily bounded by the revenue the seller would obtain by setting the deterministic supply fixed at the conditioning supply realization.

[54] In a working version of this paper, we provide a tighter bound on expected revenue, in which we average $\pi^{\delta}(Q;s)$ over the auxiliary distribution $J \equiv 1-(1-F)^{(n-1)/n}$. The bound presented in equation (3) is simpler to derive, and is sufficient for all our results.

[55] We can replace the assumption that the support of F is compact with other assumptions that guarantee that the optimal solution exists, such as for instance that there is a finite $q > 0$ such that for all s, $v(q;s) = 0$.

4.2 Full Disclosure

Our analysis also shows that the seller would like to fully reveal the realized supply: the seller thus finds transparency optimal both in the sense of setting a deterministic supply (or supply cap) and in the sense of revealing the seller's information about supply. To formalize this full-disclosure insight we enrich our base model as follows. We assume that the distribution of supply is exogenously given and commonly known. Before learning the realization of supply, the seller can publicly commit to an auction design (reserve price and supply restriction) and a disclosure policy; a disclosure policy maps the realization of supply to a distribution of public announcements (messages) from an arbitrary space of messages.[56] After committing to a disclosure policy and an auction design, the seller learns the realization of supply and publicly announces the message prescribed by the disclosure policy. Then, the bidders learn their value and bid in the auction.

Theorem 6. [Optimality of Information Disclosure] *The seller's expected revenue is maximized when the seller commits to fully reveal the realization of supply.*

Before presenting a surprisingly simple argument deriving this theorem from our preceding results, let us observe that Theorem 6 remains valid even if the seller does *not* optimize the reserve price and supply cap in the auction and these parameters of the auction are arbitrarily set, with no change in the proof. In addition, because we prove Theorem 6 for the environment in which the seller can commit to a disclosure strategy, the same full disclosure insights a fortiori holds true for environments where the seller cannot commit.

Proof. Suppose that the seller commits to a disclosure strategy and this strategy leads to a message that induces the bidders to believe that the (conditional) distribution of supply is $\hat{F}$ with upper bound of support $\hat{Q}$. The revenue bound obtained in the proof of Theorem (5) gives

$$\mathbb{E}\left[\pi^{F}\left(Q;s\right)\right] \leq \int_{0}^{\hat{Q}^{R}} \mathbb{E}_{s}\left[\pi^{\delta}\left(x;s\right)\right] d\hat{F}\left(x\right),$$

and thus expected revenue is bounded above by the expected revenue obtained by the seller fully revealing to the bidders the realization of supply. In consequence, the seller's expected revenue is maximized when the seller ex ante commits to fully reveal the realization of supply. □

[56] We maintain the global assumption of this section that all induced auction equilibria are in pure strategies; in Appendix A we show that the full disclosure insight remains valid without this assumption .

4.3 Discussion

The bound on revenue obtained in Theorems 5 and 6 applies ex ante, and that optimal (ex ante) supply or transparency may result in ex post revenue reduction. This is straightforward to see. When there is a high realization of random supply, bids for inframarginal units are above the market-clearing price (Theorem 3), and thus per-unit revenue is above the market-clearing price; at the unique realization of deterministic supply, bids for all units are equal to the market-clearing price. Then optimal pay-as-bid auctions reduce the revenue obtained when large supply is realized, while at the same time increasing expected revenue. An implication is that optimal pay-as-bid auctions improve expected revenue while decreasing the variance of revenue.

The transparency result substantially simplifies the seller's optimization problem. The problem becomes one of setting reserve price R and deterministic supply Q so as to maximize

$$\begin{aligned}\mathbb{E}_s[\pi] = &\Pr\left(v\left(\frac{Q}{n};s\right)\geq R\right)\mathbb{E}\left[v\left(\frac{Q}{n};s\right)\middle|v\left(\frac{Q}{n};s\right)\geq R\right]Q\\ &+\Pr\left(v\left(\frac{Q}{n};s\right)<R\right)R\mathbb{E}\left[nv^{-1}(R;s)\middle|v\left(\frac{Q}{n};s\right)<R\right].\end{aligned}$$

When signal s comes from an atomless distribution on a subset of $\mathbb{R}$ and the bidders' marginal values are increasing in the signal, the seller can separately maximize the reserve $R^\star$ conditional on low signals $s<\hat{s}$ and supply $Q^\star$ conditional on high signals $s\geq\hat{s}$ where $\hat{s}=\inf\{s\colon v(Q^\star/n;\hat{s})\geq R^\star\}$. The separability allows us to solve for optimal auctions. For example, suppose that the common signal s is distributed uniformly on $(\underline{s},\overline{s})$ and $v(q;s)=s-\rho q$ for some constants $\rho,\underline{s},\overline{s}>0$ such that $\overline{s}>\underline{s}\geq\rho\overline{Q}/n$. Then bidding strategies are linear, the optimal reserve price is $R^\star=\frac{\overline{s}+3\underline{s}}{8}$, optimal supply is $Q^\star=\left(\frac{3\overline{s}+\underline{s}}{8\rho}\right)n$, and the resulting expected revenue is $\frac{n}{2\rho}\left(\frac{m^2}{2}+\frac{3V}{8}\right)$ where $m=\frac{\overline{s}+\underline{s}}{2}$ is the mean and $V=\frac{(\overline{s}-\underline{s})^2}{12}$ the variance of the signal distribution. In particular, the seller revenue is increased by a mean-preserving spread of the distribution of bidders' values, a property that obtains beyond the example.[57]

[57] The monotonicity of revenue in mean-preserving spreads does not hinge on the value distribution being uniform. We provide details and related results in our supplementary note [Pycia and Woodward, 2020].

5 The Auction Design Game: Pay-as-Bid Dominates Uniform-Price

Sellers of homogenous goods are not restricted to running pay-as-bid auctions, and the uniform-price auction is the other of the two most-commonly implemented formats for of auctions of homogenous goods. From a practical perspective, which of these two formats is preferred is a an important question that has been studied both in the theoretical and empirical literature on mutli-unit auctions; see the introduction.[58] Unlike this literature—which compares the formats without taking the seller's endogenous choices into account—we explicitly model the seller's choice between the pay-as-bid and uniform-price formats, as well as among supply distributions and reserve prices, as an extensive-form game.

This auction design game has two stages. In the first stage, the seller commits to a reserve price, a distribution of supply, and the auction format (pay-as-bid or uniform-price). We also consider constrained design games in which the auction format is fixed; we refer to these as pay-as-bid design game and uniform-price design game. In the second stage, bidders participate in the specified auction.[59] We consider perfect Bayesian equilibria of these games. This structure allows us to compare outcomes of optimally designed pay-as-bid and uniform-price auctions, and to discuss the economic implications of mechanism selection. Our main insight is that choosing pay-as-bid is weakly dominant for the seller.

5.1 Revenue

We start our analysis of revenue-maximizing design with the case of uninformed seller and symmetrically informed bidders; in Section 6 we relax this assumption, allowing for asymmetrically informed bidders. For the pay-as-bid auction, Theorem 2 states that equilibrium bids are essentially unique conditional on the distribution of supply, and Theorem 5 states that optimal supply is deterministic. Together these immediately imply that equilibrium revenue is unique in the pay-as-bid design game.

Corollary 4. [Revenue in Pay-as-Bid Design Game] *In the pay-as-bid design game*

[58]From a theoretical perspective, we might be also interested in the question what general selling mechanism is optimal, but in the environments we focus on the bidders' private information is correlated and hence the seller can nearly extract their full surplus using Crémer-McLean-type mechanisms (cf. Crémer and McLean [1988] and Myerson [1981]). In the benchmark case of the current section, in which bidders' information is symmetric, full extraction of bidders' surplus is possible: e.g. the seller can ask all bidders to report their private information and set each bidder's allocation and payment in a way that fully extract the surplus of that among announced types that maximizes the seller's revenue.

[59]The bid functions $b^i(\cdot; s, R, F)$ depend on the bidders' signal as well as the auction format and the reserve prices R and supply distributions F chosen by the seller. When there is no risk of confusion, when referring to the bids on the equilibrium path we sometimes suppress the seller's choices.

with symmetrically informed bidders, the perfect Bayesian equilibrium revenue is uniquely determined and the seller can achieve it by setting optimal deterministic supply.

As noted in the discussion preceding Theorem 2, in the pay-as-bid auction equilibrium outcomes are unique but bids for never-realized quantities may not be uniquely defined. The perfect Bayesian equilibrium is thus essentially unique, and with slight abuse of terminology we also refer to it as the unique equilibrium of the pay-as-bid design game, ignoring (as in Section 3) potential multiplicity for infeasible quantities that have no impact on the observed outcomes. In this unique equilibrium bids are flat, and equal to the maximum of the reserve price R and the marginal value for the per-capita maximum, $v(Q/n; s)$.

In the uniform-price design game the analysis is more complicated. With symmetrically-informed bidders, equilibrium bids in the uniform-price auction are optimal for every realization of supply, a point first made by Klemperer and Meyer [1989]: for a given bidder, every realization of supply determines a residual supply curve corresponding to the demands of the other bidders, and the given bidder's bid effectively serves to select the price-quantity pair from this residual supply curve; this choice does not depend on choices at other realizations of supply as long as the resulting bid curve is downward-sloping. In effect, two supply distributions with the same support admit the same set of equilibria, and if one supply distribution has a smaller support than another, its set of equilibrium bids is a weak superset of the other. This implies that the revenue-maximizing equilibrium for deterministic supply is also revenue-maximizing among all possible supply distributions. In this sense, deterministic supply is also optimal in uniform-price auctions:

Lemma 1. [Deterministic Dominance in Uniform-Price Design Game] *With symmetrically-informed bidders, for any equilibrium of the uniform-price design game $((R, F), b)$, there is a deterministic-supply equilibrium $((R^\star, F^\star), b^\star(\cdot; s, R^\star, F^\star))$ that generates weakly higher seller revenue and has the same bids.*

Lemma 1 does not imply that all equilibria of the uniform-price design game have deterministic supply. Because bidders' strategies can depend on the chosen distribution of supply, it is possible that choosing deterministic supply will yield lower revenue than random supply. Consider bidders who bid the reserve price when supply is deterministic, but submit relatively aggressive bids otherwise. Then the seller could concentrate the supply distribution around the deterministic optimum, but retain some randomness to ensure that bidders submit aggressive bids. This will revenue-dominate deterministic supply, where bidders submit relatively weak bids. We develop this observation in Lemma 2 below.

The equilibria of the uniform-price game generate weakly less seller revenue than the unique equilibrium of the pay-as-bid design game, hence the pay-as-bid design game yields greater revenue than the uniform-price design game in general.

Theorem 7. [Revenue Comparison of Design Games] *With symmetrically-informed bidders, the unique equilibrium of the pay-as-bid design game generates weakly greater revenue than all equilibria of the uniform-price design game, and there is an equilibrium of the uniform-price design game that generates the same expected revenue as the unique equilibrium of the pay-as-bid design game.*

The revenue comparison is strict for all uniform-price equilibria in which bids $b^\star(\cdot; s, R^\star, F^\star)$ are strictly below the realized marginal value $v(Q^\star/n; s)$. Such equilibria are typical in the sense that in the uniform-price auction, for any Q and s, any price $p \in [R^\star, v(Q/n; s)]$ is supportable in equilibrium (see our supplementary note [Pycia and Woodward, 2020]). The proof of Theorem 7 leverages the optimality of deterministic supply established in Corollary 4 and Lemma 1 above. This major endogenous simplification allows us to show that for any signal s, the equilibrium market clearing price $p^{\star \mathrm{PAB}}(s)$ in pay-as-bid design game is weakly higher than the equilibrium market clearing price $p^{\star \mathrm{UP}}(s)$ in any equilibrium of the uniform-price design game, regardless of the level of deterministic supply. The full proof is provided in Supplementary Appendix E.

Finally, consider the unconstrained auction design game in which the designer first commits to implement a pay-as-bid or uniform-price auction, and then the selected auction format is run. Theorem 7 implies that in the auction design game, the seller either implements a pay-as-bid auction or is indifferent between the two formats because bidders bid aggressively in the uniform-price design game.

Corollary 5. [Revenue Equivalence Across Perfect Bayesian Equilibria] *All perfect Bayesian equilibria of the auction design game are revenue equivalent. Furthermore, the seller either implements a pay-as-bid auction or is indifferent between the pay-as-bid and uniform-price auctions.*

5.2 Welfare

Pay-as-bid not only generates weakly greater revenue than the uniform-price: we now show that it can generate strictly higher revenue and strictly higher payoffs for all bidder types. At the same time, the comparison of outcomes other than revenue—e.g., quantity sold, optimal reserve price, bidders' payoffs, and expected surplus—depends on the perfect Bayesian equilibrium played. The underlying reason for this dependence is the multiplicity of equilibria in uniform-price auctions. In effect, in a perfect Bayesian equilibrium of the uniform-price design game the seller can be rewarded with relatively high bids on equilibrium path and punished by lower bids off-path.[60] The argument hinges the value space being sufficiently *rich* in the following sense: there is no pair of optimal reserve and supply that maximizes ex post revenue irrespective of the bidders' signal (this assumption rules out the complete information case; for the complete information case see our supplementary note [Pycia and Woodward, 2020]).

Lemma 2. [Quantity and Reserve in Uniform Price] *Suppose the value space is rich and let $R^{\star PAB}$ and $Q^{\star PAB}$ be optimal reserve and supply in the pay-as-bid design game. There is $\varepsilon > 0$ such that for all $R^\star, Q^\star$ with $|R^{\star PAB} - R^\star| < \varepsilon$ and $|Q^{\star PAB} - Q^\star| < \varepsilon$, there is an equilibrium of the uniform-price design game in which the designer selects reserve $R^\star$ and deterministic quantity $Q^\star$.*

The proof builds on the construction of low-price robust equilibrium in our supplementary note [Pycia and Woodward, 2020], which extends the insights of Klemperer and Meyer [1989]. Two properties of this construction are important. First, bids in these robust equilibria are continuous in reserve price and do not depend on the supply distribution and thus the expected revenues are continuous in reserve prices and supply distribution (continuity in the distribution is with respect to the supremum norm on the c.d.f. representation). Second, for any deterministic $R^\star$ and $Q^\star$, the richness of values implies that the expected revenue from the low-price robust equilibrium is strictly lower than from the truthful bidding equilibrium.

The perfect Bayesian equilibrium implementing deterministic reserve $R^\star$ and quantity $Q^\star$ is then constructed as follows. If the seller sets $R^\star$ and $Q^\star$ then, in the continuation game, the bidders bid truthfully. If the seller sets different reserve or different distribution of supply then, in the continuation game, the bidders play the robust equilibrium constructed in our supplementary note Pycia and Woodward [2020]. In light of the discussion above, for small ε this continuation play incentivizes the seller to set $R^\star$ and $Q^\star$ in the first stage of the auction design game.[61]

Theorem 8. [Ambiguous Bidder Welfare Comparison] *If the value space is rich then the uniform-price design game admits perfect Bayesian equilibria in which the payoff of all bidder types is strictly higher and pefect Bayesian equilibria in which the payoff of all bidder types is strictly lower than in the unique equilibrium of the pay-as-bid design game.*

[60] In our supplementary note [Pycia and Woodward, 2020] we show that a natural equilibrium selection in the uniform-price auction provides strictly lower revenue than the optimal pay-as-bid auction. If in the dynamic game bidders employ such a low-revenue bid profile whenever a particular reserve price and quantity is *not* selected the designer can be induced to implement a particular quantity distribution and reserve price.

[61] Because truthful bids do not depend on the reserve price or the supply distribution, the expected revenue conditional on truthful bidding is continuous in reserve price and supply distribution. As expected revenue is continuous given both truthful and robust strategy profiles, and the difference between them is bounded away from zero following seller's choice of $R^{\star\mathrm{PAB}}$ and $Q^{\star\mathrm{PAB}}$, for all $R^\star$ and $Q^\star$ within small ε of $R^{\star\mathrm{PAB}}$ and $Q^{\star\mathrm{PAB}}$ (respectively) the difference is uniformly bounded away from zero.

Lemma 2 implies that there are equilibria of the uniform-price design game in which $R^{\star\mathrm{UP}} < R^{\star\mathrm{PAB}}$ and $Q^{\star\mathrm{UP}} > Q^{\star\mathrm{PAB}}$, as well as equilibria in which $R^{\star\mathrm{UP}} > R^{\star\mathrm{PAB}}$ and $Q^{\star\mathrm{UP}} < Q^{\star\mathrm{PAB}}$. The former generate higher payoff for all bidder types than the pay-as-bid design game, and the latter generate lower payoff for all bidder types than the pay-as-bid design game.[62]

Lemma 2 further implies that there are equilibria of the uniform-price design that are worse for all market participants: revenue, bidder surplus, and aggregate surplus may all be strictly lower in the uniform-price auction than in the unique equilibrium of the pay-as-bid design game.[63]

Theorem 9. [Pay-as-bid Preferred by All] *If the value space is rich then the uniform-price design game admits equilibria in which both the seller's revenue and the payoff of all bidder types is strictly lower than in the unique equilibrium of the pay-as-bid design game.*

6 Asymmetric Information among Bidders

In this final section, we relax the assumption that bidders are symmetrically informed and allow for heterogeneous signals s_i.

Recall that $s_i = (s, \theta_i)$ where s is a common signal known to all bidders and θ_i is idiosyncratic and privately known only to bidder i; notice that we do not require that θ_i and θ_j are independent nor do we require that they are identically distributed.[64] For the sake of expositional simplicity we normalize the signals so that each idiosyncratic signal θ_i has identical support containing 0, and we treat the case of all idiosyncratic signals taking value 0 as the benchmark common signal case.[65] Letting $\mathcal{S}_s = \operatorname{Supp} s$ and $\mathcal{S}_\theta = \operatorname{Supp} \theta_i$, we assume that bidder information has full support, so that $\operatorname{Supp} \theta_i|_{s,\theta_{-i}} = \mathcal{S}_\theta$, but otherwise

[62][Ausubel et al., 2014] show that both revenue and efficiency cannot be generically compared between the pay-as-bid and uniform-price auction formats, and that the comparison may vary with model specification. As they assume that the reserve price is zero and supply is unoptimized, there is neither a contradiction between the ambiguity they report and our revenue dominance, nor are our welfare comparisons implicit in theirs. Furthermore, the ambiguity we uncover is driven by equilibrium selection and Theorem 8 states that the welfare comparison is ambiguous in every model with rich values. In contrast, they provide examples of ambiguity that hinge on comparing equilibria between different model specifications and that rely on ex-ante asymmetries between bidders.

[63]Note that the uniform-price design game does not admit equilibrium in which both the seller's revenue and bidders' payoffs are higher than in the unique equilibrium of the pay-as-bid design game; see Theorem 7.

[64]The separation of signals into common and idiosyncratic components is convenient but inessential; signals can always be separated in this way and the separation simplifies the definition of bounded informational asymmetry as well as comparisons to the benchmark model with only a common signal.

[65]Our results (and the definition of bounded asymmetry) allow heterogenous $v^i(\cdot;\cdot,\cdot)$, provided $v^i(\cdot;\cdot,0)$ does not depend on i.

there are no distributional assumptions on s, θ_i, or their interrelation.

Definition 1. For $\delta \geq 0$, we say that informational asymmetry is *δ-bounded* if, for all $(s, \theta) \in \mathcal{S}_s \times \mathcal{S}_\theta$, $\sup_q |v(q; s, 0) - v(q; s, \theta)| \leq \delta$.

When marginal values v are bounded above by $\delta \geq 0$, informational asymmetries are δ-bounded. Thus δ-bounded informational asymmetry is a relatively weak restriction if δ is large; at the same time our results become tight only as δ becomes small.

For bounded asymmetries, we show that the expected revenue in any equilibrium of an optimal pay-as-bid auction—that is, pay-as-bid with optimal supply and reserve price—with asymmetric private information is *nearly above* the expected revenue in the unique equilibrium of the optimized auction when bidders' information is symmetric: expected revenue is above the revenue in the uniform price auction, decreased by $\delta Q^\star$, where $Q^\star$ is the optimal supply in the symmetric information environment (with $\theta_i = 0$). We analogously define *nearly below* and *nearly indifferent*.

Theorem 10. [A Bound on Revenue Loss from Informational Asymmetry] *Suppose that asymmetry is δ-bounded. Then, the expected revenue in any equilibrium of the optimal pay-as-bid auction is nearly above the expected revenue in the unique equilibrium of the optimal pay-as-bid auction with symmetric bidder information (in which $\theta_i = 0$).*

This theorem implies that small informational asymmetries do not dramatically reduce the seller's revenue below the symmetric-information benchmark. This implication of our theorem is not a simple limit result. First, in environments for which purification results have been proven, a limit of equilibria as we decrease the import of idiosyncratic signals is a mixed-strategy equilibrium in the limit environment, but in Theorem 10 we bound the revenue from below by a pure-strategy equilibrium in the limit environment. Second, there are so far no purification results for such infinitely-dimensional discontinuous games as pay-as-bid auctions. We are able to establish the above theorem because of our earlier results showing that when bidders' information is symmetric then optimal supply is deterministic. The following result plays a key role in its proof.

Lemma 3. *In a pay-as-bid auction with deterministic supply and reserve R, the expected revenue in any equilibrium with δ-bounded asymmetric private information is nearly above the expected revenue of pay-as-bid with same supply and reserve* $\max\{R - \delta, 0\}$ *in the unique equilibrium when bidders' information is symmetric.*

The lemma follows from Theorem 1, in which we establish that the market clearing price is bounded below by the lowest marginal value $v(\cdot; \cdot, \cdot)$ of per capita supply. When $v(\cdot; \cdot, \cdot)$

is within δ of $v(\cdot;\cdot,0)$, this lowest marginal value of per capita supply is weakly above $v\left(\frac{Q}{n};\cdot,0\right)-\delta$. By setting the deterministic quantity Q at optimal value at symmetric information and lowering the reserve price by δ with respect to optimal reserve R at symmetric information, the seller can sell at least the same quantity of the good. When bidders are asymmetrically informed, the per-unit price is bounded below by $\max\{v\left(\frac{Q}{n};s,0\right)-\delta,R-\delta\}$, while $\max\{v\left(\frac{Q}{n};s,0\right),R\}$ is the per-unit revenue in the unique equilibrium when bidders' information is symmetric and equals $(s,0)$. Thus Lemma 3 obtains.

Theorem 10 then follows from Lemma 3 because our transparency results guarantee that deterministic supply is optimal in the symmetric information benchmark, and because allowing the seller to re-optimize supply in the presence of informational asymmetries weakly improves revenue.

6.1 Pay-as-Bid vs. Uniform-Price

We now show that the revenue comparison results of Section 5 continue to hold for asymmetrically-informed bidders when the informational asymmetry is small. The comparison requires understanding the behavior of asymetrically-informed bidders in pay-as-bid auctions and uniform-price auctions as well as the design response of the seller. The logic developed above, giving an approximate revenue bound in the pay-as-bid auction when the informational asymmetry is δ-bounded, also applies to the uniform-price auction, with the exception that equilibrium may be nonunique. In the uniform-price auction with private information, every equilibrium generates revenue that is weakly below quantity times the maximum marginal value for per capita supply.

Lemma 4. *Fix a reserve price and supply distribution. In a uniform-price auction, the expected revenue in an equilibrium with asymmetrically-informed bidders is nearly below the expected revenue under truthful bidding with symmetrically-informed bidders.*

Note that revenue under truthful bidding is an upper bound on equilibrium revenue.

Proof. In the uniform-price auction, the parts of bids that determine the equilibrium market-clearing price are bounded above by truthful reporting. Let Q be a realization of supply; notice that Q is bounded above by the maximum supply $\bar{Q}$. Conditional on this supply realization, for any fixed $\delta>0$, the expected revenue under asymmetric information that is within δ of $v(\cdot;s,0)$ is bounded above by $Q\left[v\left(\frac{Q}{n};s,0\right)+\delta\right]\leq Qv\left(\frac{Q}{n};s,0\right)+\delta\bar{Q}$. The result follows because $Qv\left(\frac{Q}{n};s,0\right)$ is the revenue under truthful bidding, conditional on the common signal being s when bidders are symmetrically informed. $\square$

The above two lemmas establish an approximate version of pay-as-bid revenue dominance. With slight abuse of terminology, in the following theorem we say that the pay-as-bid expected revenue is *nearly above* uniform-price expected revenue when the difference between the two is bounded from below by $-2\delta Q^\star$ (rather than $-\delta Q^\star$ as before), where $Q^\star$ is the optimal supply in the symmetric information environment.

Theorem 11. [**Approximate Revenue Dominance of Optimal Pay-as-Bid**] *With optimal reserve price and supply, the expected revenue in the pay-as-bid auction is nearly above expected revenue in the uniform-price auction.*

In particular, for any $\varepsilon > 0$, if the informational asymmetry is $\frac{\varepsilon}{2Q^*}$-small, where Q^* is the optimal supply in the symmetric information environment, then $\mathbb{E}\left[\pi^{\mathrm{PAB}}\right] \geq \mathbb{E}\left[\pi^{\mathrm{UP}}\right] - \varepsilon$.

We show below that deterministic supply is nearly optimal, but the optimal supply does not need to be deterministic. Still, an analogue of the above theorem obtains for potentially suboptimally-designed auctions as long as they are deterministic.

Theorem 12. [**Approximate Revenue Dominance of Pay-as-Bid with Deterministic Supply**] *Given any deterministic supply Q and reserve price R, expected revenue in the pay-as-bid auction is nearly above expected revenue in the uniform-price auction. Moreover, the seller is nearly indifferent between any equilibrium of the pay-as-bid auction and any revenue-maximizing equilibrium of the uniform-price auction.*

Remark 3. The analogue of this theorem continues to hold if there is small uncertainty over supply. Without the asymmetry of information, this point follows from the continuity of optimal bidding strategies in pay-as-bid with respect to supply because our bound on uniform-price revenue is in terms of truthful bidding. The asymmetry of information does not affect the uniform-price bound, and we can control the change in the lower bound on pay-as-bid revenue via the price bound of Theorem 1.

The above two theorems tell us that, while the uniform-price auction might generate greater revenue than a pay-as-bid auction, this difference will not be large without a significant informational asymmetry among bidders or significant randomness in supply. Thus, a version of Corollary 5 holds in the presence of informational asymmetries: the seller either strictly prefers a pay-as-bid auction or is approximately indifferent between the pay-as-bid and uniform-price auctions.

6.2 Approximate Optimality of Transparency

Our analysis of elastic supply and mixed-strategy equilibria in Appendix A shows that if buyers' values are regular, a deterministic supply curve maximizes the seller's revenue. In this subsection we apply this analysis to the design of optimal pay-as-bid auctions in the presence of small informational asymmetries.

Definition 2. [Regular Demand] Let $\mathcal{S} = \{(p^\star, q^\star) \colon \exists s,\ p^\star \in \arg\max_p pv^{-1}(p;s),\ q^\star = v^{-1}(p;s)\}$ be the set of optimal monopoly price-quantity pairs. Bidder values are *regular* if, for any $(p,q),(p',q') \in \mathcal{S}$, the inequality $p' < p$ implies $q' < q$.

Values are regular if the monopolist's optimal price and quantity are in monotone correspondence.[66] When values are increasing in signal s and v^{-1} is differentiable, demand is regular when $p + v^{-1}(p;s)/v_p^{-1}(p;s)$ is increasing in s.[67] Thus our regularity condition is similar to the regularity condition in [Myerson, 1981]. When values are regular, the auctioneer can use an elastic supply curve to screen for bidder signal s, and a deterministic elastic supply curve maximizes the seller's revenue.

Theorem 13. [Approximate Optimality of Transparency] *Suppose buyers' values are regular. For any $\varepsilon > 0$, there is $\delta > 0$ such that if informational asymmetry is δ-small then there is a deterministic elastic supply curve S that is approximately optimal: $\mathbb{E}\left[\pi^S\right] \geq \mathbb{E}\left[\pi^K\right] - \varepsilon$ for any, potentially stochastic, elastic supply K.*

In Appendix A, we show that when the bidder's private information $s_i = (s, \theta_i)$ is known to the seller, the seller's revenue is strictly higher when a deterministic quantity is sold than when the buyer faces any randomness in residual supply. Then revenue with asymmetric information is bounded above by the revenue the seller would obtain with optimal monopoly supply targeted to each bidder's private information s_i.

Monopoly revenue is strictly increasing in marginal value. Then when asymmetric information is δ-small,

$$\max_q q \cdot \left[v\left(q;s,0\right) - \delta\right] \leq \max_q q \cdot v\left(q;s,\theta_i\right) \leq \max_q q \cdot \left[v\left(q;s,0\right) + \delta\right].$$

Furthermore, if the seller knows s but not θ_i, we may bound optimal expected monopoly profits below by

$$\max_q q \cdot \left[v\left(q;s,0\right) - \delta\right] \leq \max_q \mathbb{E}\left[q \cdot v\left(q;s,\theta_i\right)\right].$$

When demand is regular, it follows that expected revenue under deterministic elastic supply

[66] Recall that we do not make any assumptions on the bidders' type space, and in particular we do not require that demand increases with type.

[67] To maximize profits, $d[pv^{-1}(p;s)]/dp = 0$, implying $p + v^{-1}(p;s)/v_p^{-1}(p;s) = 0$. If the left-hand side is increasing in s, then $p^\star$ is increasing in s. To have quantity also increasing in s, we need $d[qv(q;s)]dq = 0$, or $qv_q(q;s) + v(q;s) = 0$. Under monopoly, $q = v^{-1}(p;s)$ and $p = v(q;s)$, and the conditions for monotonicity in price and in quantity are equivalent.

cannot be significantly worse than expected revenue under optimal elastic supply, where the difference is no greater than $2\delta Q^{\max}$.

6.3 Relationship to Empirical Findings

Cross-country comparisons find that pay-as-bid auctions are more than twice as prevalent as uniform-price auctions (see Brenner et al. [2009] for treasury securities; see Del Río [2017] for electricity auctions), and our results provide a theoretical explanation for the popularity of the pay-as-bid format. If revenue-interested sellers are at worst nearly indifferent between pay-as-bid and uniform-price auctions, it is natural to suspect that pay-as-bid auction should be implemented more frequently.

Corollary 5 and Theorem 11 provide an explanation of the empirical finding that revenues in pay-as-bid are close to the counterfactual revenues in uniform-price as discussed in the Introduction. The explanation is two-fold. First, by Corollary 5, the auction format is selected by the seller and a revenue-maximizing seller weakly prefers the uniform-price format only if this format is nearly equivalent to pay as bid. The South Korean Treasury auctions studied by Kang and Puller [2008] and U.S. Treasury auctions studied by Hortaçsu, Kastl, and Zhang [2018] run the uniform-price format and hence Corollary 5 provides a potential explanation of the revenue equivalence they find. Second, the optimal pay-as-bid and uniform-price auctions generate the same revenue only in the seller-optimal equilibrium of the uniform-price auction and this is precisely the equilibrium in which bids are equal to marginal values at realized quantities. The latter equality is imposed in counterfactual revenue estimation of uniform-price auctions in Hortaçsu and McAdams [2010] and Marszalec [2017] which assume truthful reporting in the uniform-price auction. Our results thus suggest that the empirical ambiguity of cross-mechanism revenue comparison is strongly tied to sellers' endogenous selection of auction format and to equilibrium selection in the empirical literature.

7 Conclusion

We have studied multi-unit auctions in an environment in which there is only limited asymmetry of information between bidders, but the seller (or auction designer) is potentially much less informed. For the limit case in which bidders' information is symmetric, we have established a mild condition for equilibrium existence as well as established equilibrium uniqueness

Paper	Data	Method	σ/μ	Conclusion
Marszalec [2017]	Poland	PAB $\to$ CF UP	0.00%	PAB > UP
Barbosa et al. [2020]	China	Controlled exp.	0.00%	PAB $\approx$ UP
Février et al. [2002]	France	PAB $\to$ CF UP	1.27%	PAB > UP
Armantier and Sbaï [2006]	France	PAB $\to$ CF UP	3.78%	UP > PAB
Umlauf [1993]	Mexico	Natural exp.	11.16%	UP > PAB
Mariño and Marszalec [2020]	Philippines	Natural exp.	17.60%	PAB > UP

Table 1: Revenue comparisons between auction formats, in comparison to the standard deviation of noncompetitive demand scaled by mean aggregate supply (Q); "CF" is "counterfactual."

and provided a tractable representation of bids.[68] We also proved that the limit equilibrium, without informational asymmetries among bidders, provides a lower bound on revenues in the presence of informational asymmetries.

We have used these results to analyze the design problem of the seller, allowing for bidders' private information. In particular, we established that revenue-maximizing pay-as-bid auctions generate more revenue than uniform-price auctions, and strictly more revenue in most cases, but welfare comparisons are inherently ambiguous. In particular, it is possible that revenue-maximizing pay-as-bid auctions are not only revenue—but also welfare—superior to uniform-price auctions.

As part of our analysis we established revenue equivalence between revenue-maximizing pay-as-bid auctions and the revenue-maximizing equilibrium of uniform-price auctions. Our revenue equivalence benchmark—which we prove both for optimally-designed auctions and for deterministic supply—provides an explanation for the empirical findings of approximate revenue equivalence between the two formats by imposing that the revenue maximizing equilibrium obtains in uniform-price auctions; this is precisely the assumption that we show leads to theoretical revenue equivalence.

Our revenue comparison and equivalence results are consistent with the second-order details of empirical findings regarding multi-unit auction revenue; Table 1 relates revenue comparisons from the literature to normalized randomness in aggregate supply, and as expected given our results its shows that for small randomness pay-as-bid and uniform price are equivalent or pay-as-bid is revenue dominant, while for larger randomness either format can be revenue dominant.[69]

[68] We hope that the tractability of our representation will stimulate future work on this important auction format. Wittwer [2017] discusses the intuition behind our representation.

[69] Table 1 summarizes all empirical studies for which we have the data allowing us to calculate the relative randomness of a single run of an auction. The randomness measures σ/μ are taken from either published work

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A Elastic Supply

In the main text we (mostly) focus on pure strategy-equilibria and on designing a potentially stochastic supply distribution allowing for a separately set reserve price. Our essential insights remain valid if we allow mixed-strategy equilibria and potentially stochastic elastic supply curves.[70]

We study the seller who selects a distribution over reserve prices, possibly correlated with the distribution of quantity. Let $K(Q;R)$ be a *supply-reserve distribution*, giving the probability that realized quantity is $\tilde{Q} \le Q$ or the realized reserve price is $\tilde{R} > R$,[71]

$$K(Q;R) = \Pr\left(\tilde{Q} \le Q\right) + \Pr\left(\tilde{Q} > Q, \tilde{R} > R\right).$$

Note that conditional on aggregate demand $p(\cdot)$, $K(Q;p(Q))$ is the probability that realized aggregate supply is below Q: either realized supply is $\tilde{Q} \le Q$, or realized reserve is $\tilde{R} > p(Q)$ and quantity is constrained. The following special cases illustrate the supply-reserve distribution K:

- If K is equivalent to a random supply distribution F then $K(Q,R) = F(Q)$;
- If K is equivalent to a random reserve distribution F^R then $K(Q,R) = 1 - F^R(R)$;
- If K is equivalent to deterministic supply curve S then $K(Q,R) = 1[S(R) < Q]$.

We allow an arbitrary distribution of the bidders' common signal s. The idiosyncratic signals θ_i are assumed to be 0 and dropped from notation.[72]

[70] As discussed in footnote 58, in the special case of our model with perfectly correlated types, $(s,\theta_i) = (s,0)$, the seller can fully extract the bidders' rents. Furthermore, in our model of asymmetrically informed bidders with demand curves coming from δ support around commonly known (among bidders) demand, the seller could come within order-δ of full extraction: even if idiosyncratic information is i.i.d., the seller can extract from the bidders the common part of the demand curve.

[71] In general, $K(Q;R) = 1 - \Pr(\tilde{Q} > Q, \tilde{R} < R)$, and K is not a cumulative distribution function. In the absence of mass points, however, $\Pr(\tilde{Q} \le Q, \tilde{R} \le R) = K(Q;R) - K(0;R)$, and the cumulative distribution function is in one-to-one correspondence with K.

[72] Most of the auxiliary analysis in the accompanying Appendix G does not hinge on this assumption.

To key to extending our results to this environment—and hence to environment with idiosyncratic signals, see Section 6—is establishing the analogues of our uniqueness and optimality of deterministic supply results. Equilibrium uniqueness obtains when the elastic supply curve is deterministic because an analogue of Theorem 1 obtains (see Appendix G for details of this and other proofs).

Theorem 14. **[Unique Pay-as-Bid Equilibrium]** *If the elastic supply is deterministic then the pay-as-bid auction admits an essentially unique mixed-strategy equilibrium.*

In this essentially unique mixed-strategy equilibrium, all bidders bid their marginal value on the last allocated unit for all units they receive; they can randomize over their bids on units the do not receive with no impact on equilibrium outcome.

Perhaps paradoxically, the main difficulty in proving the optimality of deterministic elastic supply lies in establishing this result for the case when the bidders' common signal, s, is known to the seller—that is when it takes a constant value with probability 1.

Lemma 5. *Suppose bidders are symmetric and their information is known to the seller. Given any supply-reserve distribution K, there is a deterministic quantity $Q^\star$ such that the pay-as-bid auction with fixed supply $Q^\star$ raises greater revenue than the pay-as-bid auction with supply-reserve distribution K.*

We prove this auxiliary complete-information result by studying an auxiliary problem in which a bidder's bid satisfies a best-response first order condition but is not necessarily a best response to the random elastic supply and other bidders' mixed strategies. We show that if—counterfactually—the seller was able to set the random supply-reserve distribution separately for this focal bidder, holding the other bidders' behavior fixed, then the seller would optimize this part of the revenue by keeping the quantity allocated to the focal bidder constant and randomizing only over reserve prices.[73] That is, analyzing constant supply and random reserve decouples the focal bidder's best response from strategies of other bidders. Thus—given the symmetry of the problem—the seller is able to implement such a revenue maximizing scheme via a pay-as-bid auction with fixed supply and the same random supply distribution for all bidders. Leveraging the simplification brought by being able to restrict attention on random reserve only, we bound the maximum revenue of the seller by the revenue from a deterministic supply and reserve pay-as-bid (and uniform-price with identical supply and reserve).

Having shown that if the seller knew bidders' common information, then she can do no better than set deterministic elastic supply so as to maximize the revenue, it remains to

[73] We also show a further technical property that—with arbitrarily small revenue loss—the reserve distribution can be so chosen that the focal bidder submits a strictly decreasing bid.

observe that regularity (defined in Section 6) allows the seller to implement such an optimal reserve and quantity by an elastic supply function even though the seller does not know the bidders' information.

Theorem 15. **[Deterministic Auctions Are Optimal]** *When bidder values are regular then revenue in the pay-as-bid auction is maximized by implementing a deterministic supply curve. Any mixed-strategy equilibrium of the pay-as-bid auction with any random elastic supply raises weakly lower revenue than the unique equilibrium of pay-as-bid with optimal deterministic supply.*

Because deterministic elastic supply is not only optimal in pay-as-bid, but also extracts the same revenue as if the seller knew bidders' values, we can also conclude the following:

Theorem 16. **[Pay-as-Bid Revenue Dominance]** *If bidder values are regular then the unique equilibrium of the optimal pay-as-bid auction raises weakly more revenue than any equilibrium in uniform-price auction with any supply-reserve distribution.*

Furthermore, for a generic distribution of values there are multiple equilibria in uniform-price, and the revenue in a generic uniform-price equilibrium is strictly lower than the revenue in optimal pay-as-bid. This last point follows from the underpricing equilibrium constructions in, e.g.. Back and Zender [1993] and LiCalzi and Pavan [2005].

Finally, our analysis of optimal elastic supply implies that an analogue of the information disclosure Theorem 6 remains true in under random elastic supply. Recall that in this theorem the quantity is exogenously realized and the seller has the ability to communicate this cap to the bidders. Because the optimal elastic supply is constructed point-by-point and hence does not depend on the quantity cap other than in the inelastic part of the supply when the cap is binding, in the current elastic supply setting the seller still wants to set the elastic supply (where possible) and fully reveal their private information.

Theorem 17. **[Optimality of Information Disclosure with Elastic Supply]** *If the bidders' values are regular then the seller's expected revenue is maximized when the seller commits to fully reveal the realization of the elastic supply curve.*

Supplementary Appendix (For Online Publication): Proofs

B Proof of Theorem 1 and Auxiliary Lemmas

In what follows, we denote the inverse hazard rate of aggregate supply by $H = \frac{1-F}{f}$.

B.1 Proof of Theorem 1 (Bound on Market Price)

Our equilibrium analysis relies on the identification of the minimum equilibrium market clearing price. In this appendix we prove Theorem 1, which bounds this price. The arguments do not depend on the presence (or absence) of idiosyncratic private information or mixed strategies. We consolidate all bidder-known uncertainty into $\zeta_i = (s, \theta_i, \xi_i)$, where s is the signal observed by all bidders, θ_i is bidder i's idiosyncratic private information, and ξ_i is a term parameterizing bidder i's potentially-mixed strategy; thus bidder i's bid $b^i : [0, \overline{Q}] \times \operatorname{Supp} \zeta_i \to \mathbb{R}_+$.[74] Where useful, we consider $\zeta_i | s$ to hold fixed the common signal s while letting θ_i and ξ_i vary.

We also introduce notation for the (essential) minimum market clearing price $\underline{p}$ and (essential) maximum receivable quantity $\overline{q}^i$, conditional on strategy profile $(b^j)_{j=1}^n$,

$$\underline{p}(s) = \operatorname*{ess\,inf}_{Q,\zeta|s} p\left(Q; \left(b^j(\cdot;\zeta_j)\right)_{j=1}^n\right);$$
$$\overline{q}^i(\zeta_i) = \operatorname*{ess\,sup}_{Q,\zeta_{-i}|s} q^i\left(Q; b^i(\cdot;\zeta_i), b^{-i}(\cdot,\zeta_{-i})\right).$$

Thus, when the bidding strategy profile is $(b^j)_{j=1}^n$, the market clearing price is almost never below $\underline{p}(s)$ when the common signal is s, and bidder i's allocation is almost never above $\overline{q}^i(\zeta_i)$ when her type is ζ_i.

Lemma 6. *In any equilibrium, conditional on common signal s, at least $n-1$ bidders, with probability 1, bid their true value for their maximum receivable quantity. That is,*

$$\#\left\{i\colon \Pr\left(b^i\left(\overline{q}^i(\zeta);\zeta\right) = v\left(\overline{q}^i(\zeta);\zeta\right)\middle| s\right) = 1\middle| s\right\} \geq n-1.$$

Proof. For a given agent i, common signal s, and $\lambda > 0$, consider an alternative bidding

[74] For compactness we also write $v(\cdot;\zeta_i) = v(\cdot; s, \theta_i)$, but we do not imply that a bidder's marginal value may vary with her action selection from a mixed strategy.

strategy b^λ defined by

$$b^\lambda (q;\zeta_i) = \begin{cases} b^i (q;\zeta_i) & \text{if } b^i (q;\zeta_i) \geq b^i \left(\overline{q}^i (\zeta_i);\zeta_i\right) + \lambda, \\ \min \left\{ b^i \left(\overline{q}^i (\zeta_i);\zeta_i\right) + \lambda, v(q;\zeta_i) \right\} & \text{otherwise.} \end{cases}$$

Since $b^i(\cdot;\zeta_i)$ is left-continuous, for small λ this deviation will award the agent all excess quantity above $\sum_{j\neq i} \varphi^j(b^i(\overline{q}^i(\zeta_i);\zeta_i) + \lambda;\zeta_j)$. Let $q^\star(\lambda;\zeta)$ be the quantity obtained under this deviation when, under the original strategy, $q^i(\zeta)$ units would be obtained. Explicitly,

$$q^\star (\lambda;\zeta) = Q - \sum_{j\neq i} \varphi^i \left(b^i \left(\overline{q}^i (\zeta_i);\zeta_i\right) + \lambda;\zeta_j \right) = Q - \sum_{j\neq i} \underline{q}^{ji} (\lambda;\zeta),$$

where $\underline{q}^{ji}(\lambda;\zeta) = \varphi^j(b^i(\overline{q}^i(\zeta_i);\zeta_i) + \lambda;\zeta_j)$ is the quantity bidder j receives when the aggregate signal profile is ζ and bidder i implements bid b^λ; note that $\underline{q}^{ii}(\lambda;\zeta)$ is the maximum quantity for which bidder i bids above $b^i(\overline{q}(\zeta_i);\zeta_i) + \lambda$, which does not depend on ζ_{-i}, and denote this quantity by $\underline{q}^i_\lambda(\zeta_i)$. We will use the quantity $q^\star(\lambda;\zeta)$ to analyze the additional quantity the deviation yields above baseline,

$$\begin{aligned} &\Delta^i_L (\lambda;\zeta) = q^i (\zeta) - \underline{q}^{ii} (\lambda;\zeta), \quad \Delta^i_R (\lambda;\zeta) == q^\star (\lambda;\zeta) - q^i (\zeta), \\ &\Delta^i (\lambda;\zeta) = \Delta^i_L (\lambda;\zeta) + \Delta^i_R (\lambda;\zeta). \end{aligned}$$

Incentive compatibility requires that this deviation cannot be profitable, hence the additional costs must outweigh the additional benefits,

$$\begin{aligned} &\mathbb{E}_{Q,\zeta|s} \left[\int_{\underline{q}^i_\lambda(\zeta_i)}^{q^i(\zeta)} b^\lambda (x;\zeta_i) - b^i (x;\zeta_i)\, dx \middle| q_i \geq \underline{q}^i_\lambda (\zeta_i) \right] \\ &\quad \geq \mathbb{E}_{Q,\zeta|s} \left[\int_{q^i(\zeta)}^{q^\star(\lambda;\zeta)} v (x;\zeta_i) - b^\lambda (x;\zeta_i)\, dx \middle| q_i \geq \underline{q}^i_\lambda (\zeta_i) \right]. \end{aligned}$$

Importantly, this inequality must hold both ex ante and interim, unconditional on θ_i. Because bids are weakly decreasing, the left-hand expectation is bounded above by

$$\begin{aligned} &\mathbb{E}_{Q,\zeta|s} \left[\int_{\underline{q}^i_\lambda(\zeta_i)}^{q^i(\zeta)} b^\lambda (x;\zeta_i) - b^i (x;\zeta_i)\, dx \middle| q_i \geq \underline{q}^i_\lambda (\zeta) \right] \\ &\leq \mathbb{E}_{Q,\zeta|s} \left[\int_{\underline{q}^i_\lambda(\zeta_i)}^{q^i(\zeta)} b^i \left(\overline{q}^i (\zeta_i);\zeta_i\right) + \lambda - b^i \left(\overline{q}^i (\zeta_i);\zeta_i\right) \middle| q_i \geq \underline{q}^i_\lambda (\zeta_i) \right] \\ &= \lambda \mathbb{E}_{Q,\zeta_{-i}|s} \left[\Delta^i_L (\lambda;\zeta) \middle| q_i \geq \underline{q}^i_\lambda (\zeta_i) \right]. \end{aligned}$$

As marginal values are Lipschitz in quantity and $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v^i(\overline{q}^i(\zeta_i);\zeta_i)$ by assumption, the right-hand expectation is bounded above by (M is the Lipschitz modulus of v)

$$
\begin{aligned}
&\mathbb{E}_{Q,\zeta|s}\left[\int_{q^i(\zeta)}^{q^\star(\lambda;\zeta)} v\left(x;\zeta_i\right) - b^\lambda\left(x;\zeta_i\right)dx\middle| q_i \geq \underline{q}_\lambda^i\left(\zeta_i\right)\right] \\
&\quad\geq \mathbb{E}_{Q,\zeta|s}\left[\int_{q^i(\zeta)}^{q^\star(\lambda,\zeta)} \left(v^i\left(\overline{q}^i\left(\zeta_i\right);\zeta_i\right) - \left(x - q^i\left(\zeta\right)\right)M - \left(b^i\left(\overline{q}^i\left(\zeta_i\right);\zeta_i\right) + \lambda\right)\right)_+ dx\middle| q_i \geq \underline{q}_\lambda^i\left(\zeta_i\right)\right] \\
&\quad\geq \mathbb{E}_{Q,\zeta|s}\left[\frac{1}{2}\left(\mu\left(\zeta_i\right) - \lambda\right)\min\left\{\Delta_R^i\left(\lambda;\zeta\right), \frac{\mu\left(\zeta_i\right) - \lambda}{M}\right\}\middle| q_i \geq \underline{q}_\lambda^i\left(\zeta_i\right)\right],
\end{aligned}
$$

where $\mu(\zeta_i) = v^i(\overline{q}^i(\zeta_i);\zeta_i) - b^i(\overline{q}^i(\zeta_i);\zeta_i)$. If it is the case that $(\mu(\zeta_i) - \lambda)/M \leq \Delta_R^i(\lambda;\zeta)$ for all λ, then it is impossible that the overall inequality is satisfied for all λ (its left-hand side converges to zero in λ, while the right-hand side converges to a strictly positive value) and incentive compatibility is violated. Therefore we assume that the $\min\{\cdot,\cdot\}$ resolves to $\Delta_R^i(\lambda;\zeta)$. Then the overall inequality implies

$$
\lambda\mathbb{E}_{Q,\zeta|s}\left[\Delta_L^i\left(\lambda;\zeta\right)\middle| q_i \geq \underline{q}_\lambda^i\left(\zeta_i\right)\right] \geq \mathbb{E}_{Q,\zeta|s}\left[\frac{1}{2}\left(\mu\left(\zeta_i\right) - \lambda\right)\Delta_R^i\left(\lambda;\zeta\right)\middle| q_i \geq \underline{q}_\lambda^i\left(\zeta_i\right)\right].
$$

Since $\Delta_R^i(\lambda;\zeta)$ is bounded, there is $m^i(\lambda)$ such that

$$
\lambda\mathbb{E}_{Q,\zeta|s}\left[\Delta_L^i\left(\lambda;\zeta\right)\middle| q_i \geq \underline{q}_\lambda^i\left(\zeta_i\right)\right] \geq \frac{1}{2}\left(m^i\left(\lambda\right) - \lambda\right)\mathbb{E}_{Q,\zeta|s}\left[\Delta_R^i\left(\lambda;\zeta\right)\middle| q_i \geq \underline{q}_\lambda^i\left(\zeta_i\right)\right].
$$

For any i, any λ, and any $\kappa > 0$, there is $\Lambda^i(\lambda,\kappa) > 0$ such that

$$
\Lambda^i\left(\lambda,\kappa\right) < \frac{1}{2}\left(m^i\left(\lambda\right) - \lambda\right)\kappa.
$$

The term $m^i(\lambda)$ can be specified so that $m^i(\lambda)-\lambda$ is decreasing in λ, so if $\Lambda^i(\lambda;\kappa) < (m^i(\lambda)-\lambda)\kappa/2$, then $\Lambda^i(\lambda,\kappa) < (m^i(\lambda')-\lambda')\kappa/2$ for all $\lambda' > \lambda$. Then let $\overline{\Lambda} = \min\{\Lambda^i(\lambda,\kappa)\colon \Pr_{\zeta_i}(b^i(\overline{q}^i(\zeta_i);\zeta_i) > v^i(\overline{q}^i(\zeta_i);\zeta_i)|s) > 0\}$. For any such κ, $\overline{\Lambda}$, it must be that

$$
\kappa\mathbb{E}_{Q,\zeta|s}\left[\Delta_L^i\left(\overline{\Lambda};\zeta\right)\middle| q_i \geq \underline{q}_\lambda^i\left(\zeta_i\right)\right] \geq \mathbb{E}_{Q,\zeta|s}\left[\Delta_R^i\left(\overline{\Lambda};\zeta\right)\middle| q_i \geq \underline{q}_\lambda^i\left(\zeta_i\right)\right].
$$

Define bidder j with type ζ_j to be *relevant* given price p (and common signal s) if $b^j(\overline{q}^j(\zeta_j);\zeta_j) \leq p < v^j(\overline{q}^j(\zeta_j);\zeta_j)$. Fixing price p and summing the above incentive inequality over all relevant agents gives

$$\begin{aligned}
&\kappa \sum_{j \text{ relevant}} \mathbb{E}_{Q,\zeta|s}\left[\Delta_L^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right] \\
&\geq \sum_{j \text{ relevant}} \mathbb{E}_{Q,\zeta|s}\left[\Delta_R^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right] \\
&= \sum_{j \text{ relevant}} \mathbb{E}_{Q,\zeta|s}\left[\Delta^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right] - \mathbb{E}_{Q,\zeta|s}\left[\Delta_L^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right].
\end{aligned}$$

Thus,

$$(\kappa+1) \sum_{j \text{ relevant}} \mathbb{E}_{Q,\zeta|s}\left[\Delta_L^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right] \geq \sum_{j \text{ relevant}} \mathbb{E}_{Q,\zeta|s}\left[\Delta^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right].$$

By definition, $\Delta^j(\overline{\Lambda};\zeta) = Q - \underline{q}_\lambda^j(\zeta_j) - \sum_{k\neq j} \underline{q}^{kj}(\overline{\Lambda};\zeta) \equiv Q - \underline{Q}^j(\overline{\Lambda};\zeta)$ and $\Delta_L^j(\overline{\Lambda};\zeta) = q^j(\zeta) - \underline{q}_\lambda^j(\overline{\Lambda};\zeta)$. Furthermore,

$$\sum_{j \text{ relevant}} q^j\left(\zeta\right) - \underline{q}_\lambda^j\left(\zeta_j\right) \leq \sum_j q^j\left(\zeta\right) - \underline{q}_\lambda^j\left(\zeta_j\right) = Q - \underline{Q}\left(\underline{p}+\delta\right).$$

Then it follows that

$$\kappa + 1 \geq \#\left\{j \text{ relevant}\right\}.$$

Since $\kappa > 0$ may be arbitrarily small, it follows that there is at most one relevant bidder; i.e., there is at most a single bidder i such that $\Pr(b^i(\overline{q}^i(\zeta);\zeta) < v(\overline{q}^i(\zeta);\zeta)) < 1$. $\square$

Lemma 7. *For all bidders i and all bidder-common signals s,*

$$\Pr\left(b^i\left(\overline{q}^i\left(\zeta_i\right);\zeta_i\right) = v\left(\overline{q}^i\left(\zeta_i\right);\zeta_i\right)\middle|s\right) = 1.$$

Proof. Fix a common signal s. Lemma 6 shows that at least $n-1$ bidders j are such that $b^j(\overline{q}^j(\zeta_j);\zeta_j) = v(\overline{q}^j(\zeta_j);\zeta_j)$ with probability 1. If all n bidders' bids satisfy this condition, the desired result follows immediately from market clearing. Otherwise, there is some bidder i such that $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$ with $\zeta_i|s$-strictly positive probability. We show that (i) this bidder's bid must be constant in a neighborhood of $\overline{q}^i(\zeta_i)$, (ii) with $\zeta_{-i}|s$-positive probability, opposing bidders' bids are asymptotically flat near $\overline{q}^j(\zeta_j)$, and (iii) this implies that bidder i has a strict incentive to increase her (flat) bid near $\overline{q}^i(\zeta_i)$.

Let bidder i and parameter ζ_i be such that $b^i(\overline{q}^i(\zeta_i);\zeta_i) = \underline{p} < v(\overline{q}^i(\zeta_i);\zeta_i)$, and assume that b^i is strictly decreasing in a neighborhood to the left of $\overline{q}^i(\zeta_i)$. For $\lambda > 0$, define an alternate bid b^λ,

$$b^\lambda(q) = \begin{cases} b^i(q;\zeta_i) & \text{if } b^i(q;\zeta_i) \geq \underline{p}+\lambda, \\ \underline{p}+\lambda & \text{otherwise.} \end{cases}$$

Since $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$ and we analyze small $\lambda > 0$, we may assume that λ is small enough that for any feasible quantity q, $b^\lambda(q) \leq v(q;\zeta_i)$. Then whenever the market clearing price would be $p < \underline{p}+\lambda$ if bidder i submitted bid b^i, the market clearing price will be $\underline{p}+\lambda$ if she submits bid b^λ instead. Further, bidder i receives the full residual supply,

$$q_i^\lambda = Q - \sum_{j\neq i} \overline{\varphi}^j\left(\underline{p}+\lambda;\zeta_j\right).$$

The utility gain associated with bid b^λ versus bid b^i is bounded below by

$$\mathbb{E}_{Q,\zeta_{-i}}\left[\left.\begin{array}{l}\displaystyle\int_q^{Q-\sum_{j\neq i}\overline{\varphi}^j\left(\underline{p}+\lambda;\zeta_j\right)} v(x;\zeta_i) - \left(\underline{p}+\lambda\right)dx \\ \displaystyle -\int_{\overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)}^{q} \left(\underline{p}+\lambda\right) - b^i(x;\zeta_i)\,dx\end{array}\right| q \geq \overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)\right]. \tag{4}$$

Because bidder i's opponents all have $\Pr(b^j(\overline{q}^j(\zeta_j);\zeta_j) = v(\overline{q}^j(\zeta_j);\zeta_j)) = 1$, and bids are below values and values are Lipschitz continuous, there is $M > 0$ such that $\overline{q}^j(\zeta_j) - \overline{\varphi}^j(\underline{p}+\lambda;\zeta_j) > M\lambda$ with probability 1 for all $j \neq i$. Then, letting $\lambda < v(\overline{q}^i(\zeta_i);\zeta_i) - b^i(\overline{q}^i(\zeta_i);\zeta_i)$, the bound in 4 is in turn bounded below by

$$\begin{aligned}
&\mathbb{E}_{Q,\zeta_{-i}|s}\left[\left.\int_q^{\left(Q-\sum_{j\neq i}\overline{q}^j(\zeta_j)\right)+(n-1)M\lambda} v(x) - \left(\underline{p}+\lambda\right)dx - \left(q - \overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)\right)\lambda\right| q \geq \overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)\right] \\
&\geq \mathbb{E}_{Q,\zeta_{-i}|s}\left[\left.\left(\left[\left(Q - \sum_{j\neq i}\overline{q}^j(\zeta_j)\right) + (n-1)M\lambda\right] - q\right)\lambda - \left(\left[Q-\overline{Q}\right] + (n-1)M\lambda\right)\lambda\right| q \geq \overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)\right] \\
&= \mathbb{E}_{Q,\zeta_{-i}|s}\left[\overline{Q} - \sum_{j\neq i}\overline{q}^j(\zeta_j) - q\left| q \geq \overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)\right.\right]\lambda > 0.
\end{aligned}$$

In the above we rely on the fact that the minimum market clearing price is obtained when aggregate supply is maximized. Since b^λ yields higher expected utility than b^i when $\lambda > 0$ is small, b^i is not a best response, and therefore any best response b^i must be constant in a neighborhood of $\overline{q}^i(\zeta_i)$, if $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$.

Define $\check{q}^i(\zeta_i) = \underline{\varphi}^i(\underline{p};\zeta_i)$ to be the left endpoint of the flat interval of bidder i's bid, containing $\overline{q}^i(\zeta_i)$. Without loss of generality, we may assume that $b^i(q;\zeta_i) = \underline{p}$ for all $q >$

$\check{q}^i(\zeta_i)$ whenever $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$: extending the flat portion of the bid function either does not affect allocation, or (by market clearing) increases allocation to some q such that $v(q;\zeta_i) > \underline{p}$. Since $\Pr(b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)|s) > 0$ and $\check{q}^i(\zeta_i) < \overline{q}^i(\zeta_i)$ for all ζ_i with $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$, it follows that $\Pr(p(Q,\zeta) = \underline{p}|s) > 0$. Consider a bidder $j \neq i$ and type ζ_j such that $b^j(\overline{q}^j(\zeta_j);\zeta_j) = v(\overline{q}^j(\zeta_j);\zeta_j) = \underline{p}$; since $\Pr(p(Q,\zeta) = \underline{p}|s) > 0$, it must be that $\Pr(q_j = \overline{q}^j(\zeta_j)|s) > 0$. If the bid $b^j(\cdot;\zeta_j)$ is optimal, it must not be utility-improving to decrease the bid to $b^{\lambda\mu}$, where[75]

$$b^{\lambda\mu}(q) = \begin{cases} b^j(q;\zeta_j) & \text{if } q < \overline{q}^j(\zeta_j) - \lambda, \\ \underline{p} + \mu & \text{otherwise.} \end{cases}$$

The bid $b^{\lambda\mu}$ saves payment $\int_{\overline{q}^j(\zeta_j)-\lambda}^{\overline{q}^j(\zeta_j)} b^j(q;\zeta_j) - (\underline{p}+\mu)dq$ whenever $q_j = \overline{q}^j(\zeta_j)$, but potentially reduces quantity when $q_j \in (\overline{q}^j(\zeta_j)-\lambda, \overline{q}^j(\zeta_j))$. The change in utility from implementing bid $b^{\lambda\mu}$ instead of bid $b^j(\cdot;\zeta_j)$ is bounded below by

$$\begin{aligned} &\int_{\overline{q}^j(\zeta_j)-\lambda}^{\overline{q}^j(\zeta_j)} b^j(q;\zeta_j) - \left(\underline{p}+\mu\right) dq \Pr\left(q_j = \overline{q}^j(\zeta_j)\middle|s\right) \\ &- \int_{\overline{q}^j(\zeta_j)-\lambda}^{\overline{q}^j(\zeta_j)} \int_{\overline{q}^j(\zeta_j)-\lambda}^{q} v(x;\zeta_j) - b^j(x;\zeta_j)\, dx dG^i\left(q; b^j\right). \end{aligned}$$

The derivative of this expression with respect to λ must be weakly negative,

$$\begin{aligned} &\left(b^j\left(\overline{q}^j(\zeta_j)-\lambda;\zeta_j\right) - \left(\underline{p}+\mu\right)\right)\Pr\left(q_j = \overline{q}^j(\zeta_j)\middle|s\right) \\ &- \left(v\left(\overline{q}^j(\zeta_j)-\lambda;\zeta_j\right) - b^j\left(\overline{q}^j(\zeta_j)-\lambda;\zeta_j\right)\right)\Pr\left(q_j \in \left(\overline{q}^j(\zeta_j)-\lambda, \overline{q}^j(\zeta_j)\right)\middle|s\right) \leq 0. \end{aligned}$$

This inequality holds for all $\mu > 0$. Letting M be the Lipschitz modulus of v, substituting in for $b^j(\overline{q}^j(\zeta_j);\zeta_j) = \underline{p}$ means that the previous inequality implies

$$\begin{aligned} &\left(b^j\left(\overline{q}^j(\zeta_j)-\lambda;\zeta_j\right) - \underline{p}\right)\Pr\left(q_j = \overline{q}^j(\zeta_j)\middle|s\right) - M\lambda\Pr\left(q_j \in \left(\overline{q}^j(\zeta_j)-\lambda, \overline{q}^j(\zeta_j)\right)\middle|s\right) \leq 0 \\ \iff \quad & -\frac{b^j\left(\overline{q}^j(\zeta_j);\zeta_j\right) - b^j\left(\overline{q}^j(\zeta_j)-\lambda;\zeta_j\right)}{\lambda} \leq \frac{M\Pr\left(q_j \in \left(\overline{q}^j(\zeta_j)-\lambda, \overline{q}^j(\zeta_j)\right)\middle|s\right)}{\Pr\left(q_j = \overline{q}^j(\zeta_j)\middle|s\right)}. \end{aligned}$$

Taking the limit as $\lambda \searrow 0$, we obtain that $b^j_q(\overline{q}^j(\zeta_j);\zeta_j) = 0$. Thus any bidder $j \neq i$ with type ζ_j such that $b^j(\overline{q}^j(\zeta_j);\zeta_j) = v(\overline{q}^j(\zeta_j);\zeta_j)$ and $\Pr(q_j = \overline{q}^j(\zeta_j)|s) > 0$ is such that $b^j_q(\overline{q}^j(\zeta_j);\zeta_j) = 0$.

[75] The μ term ensures that bidder j wins ties against the flat portion of bidder i's bid; this term will be taken to zero and thus will have no marginal effect on utility.

Now return to bidder i with type ζ_i such that $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$ and $\check{q}^i(\zeta_i) < \overline{q}^i(\zeta_i)$, and consider the alternate bid function b^λ defined in the first portion of this proof. We now place a slightly different bound on the utility gained by implementing bid b^λ versus bid $b^i(\cdot;\zeta_i)$. Payments increase by at most $\overline{Q}\lambda$, with at most probability 1; and, whenever $q_i > \check{q}^i(\zeta_i)$ under $b^i(\cdot;\zeta_i)$, bidder i receives the full residual quantity $Q - \sum_{j\neq i}\overline{\varphi}^j(\underline{p}+\lambda;\zeta_j)$. Then a lower bound on the utility improvement generated by the alternate bid b^λ (versus $b^i(\cdot;\zeta_i)$) is

$$\mathbb{E}_{Q,\zeta_{-i}}\left[\int_q^{Q-\sum_{j\neq i}\overline{\varphi}^j\left(\underline{p}+\lambda;\zeta_j\right)} v\left(x;\zeta_i\right) - \underline{p}dx - \overline{Q}\lambda \middle| q \geq \check{q}^i\left(\zeta_i\right)\right].$$

For b^λ to not be utility-improving, this expectation must be weakly negative. Dividing through by λ and taking the limit at $\lambda \searrow 0$ gives

$$\mathbb{E}_{Q,\zeta_{-i}}\left[-\left(v\left(Q-\sum_{j\neq i}\overline{q}^j\left(\zeta_j\right);\zeta_i\right)-\underline{p}\right)\sum_{j\neq i}\overline{\varphi}_p^j\left(\underline{p};\zeta_j\right)-\overline{Q}\middle| q\geq\check{q}^i\left(\zeta_i\right)\right]\leq 0.$$

By assumption, $v(Q-\sum_{j\neq i}\overline{q}^j(\zeta_j);\zeta_j) > \underline{p}$, and from the previous paragraph we have that $\overline{\varphi}_p^j(\underline{p};\zeta_j) = -\infty$ with strictly positive probability. Then the above inequality cannot be satisfied. It follows that there is no bidder i such that $\Pr(b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)|s) > 0$. $\square$

Lemma 7 states that, for all bidders i, at the maximum quantity received with positive probability, the equilibrium bid must exactly equal the bidder's true marginal value. The inequality in Theorem 1, is

$$\min_i \operatorname*{ess\,inf}_{\tilde{\theta}_i} v^i\left(\frac{1}{n}\overline{Q}^R;s,\tilde{\theta}_i\right) \leq p\left(\overline{Q}^R;s,\theta\right) \leq \max_i \operatorname*{ess\,sup}_{\tilde{\theta}_i} v^i\left(\frac{1}{n}\overline{Q}^R;s,\tilde{\theta}_i\right).$$

On the left-hand side, there is some bidder for whom the maximum feasible quantity is no more than $\overline{Q}^R/n$. Since values are decreasing in bids, this bidder's value for $\overline{Q}^R/n$ is weakly below their value for their maximum feasible quantity, and therefore this is a lower bound for the market clearing price. The same argument applies with regard to a bidder whose maximum feasible quantity is at least $\overline{Q}^R/n$, establishing the right-hand inequality.

When bidders are symmetric and symmetrically-informed, Theorem 1 immediately implies the following result, which we use in the subsequent proofs.

Corollary 6. *When bidders have symmetric information, $(s,\theta_i) = (s,0)$ for all bidders i, the equilibrium minimum market clearing price equals the marginal value for the per-capita* *maximum quantity,*

$$\underline{p}(s) = \hat{v}\left(\overline{Q}^R; s\right).$$

B.2 Pure strategy equilibrium derivation with symmetric bidder information

In this section we present the lemmas for our results on existence, uniqueness, and bid representation of pure strategy equilibria under symmetric bidder information. To simplify notation we will thus write $v(q)$ in lieu of $v(q; s, \theta_i)$ and $b^i(q)$ in lieu of $b^i(q; s, \theta_i)$.

Let us fix a pure-strategy candidate equilibrium $(b^i)_{i=1}^n$. Recall that bid functions are weakly decreasing and (where useful) we may assume that they are right-continuous. Given equilibrium bids the market price (that is, the stop-out price) $p(Q)$ is a function of realized supply Q. In line with Appendix A, denote $G^i(q; b^i) = \Pr(q^i \leq q | b^i)$; that is, $G^i(q; b^i)$ is the probability that agent i receives at most quantity q when submitting bid b^i in the equilibrium considered. The monotonicity of bid functions implies that as long as b^i is an equilibrium bid, and given other equilibrium bids, the probability $G^i(q; b^i)$ depends on b^i only through the value of $b^i(q)$.

Our statements in the following results are generally about relevant quantities, such that $G^i(q; b^i) < 1$. For each bidder we ignore quantities larger than the maximum quantity this bidder can obtain in equilibrium; for instance, in the following lemmas, all bidders could submit identical flat bids above their values for units they never obtain. Accordingly, we restrict attention to *relevant price levels* p, such that $\Pr(p^\star < p) > 0$.

Lemma 8. *For no relevant price level p are there two or more bidders who, in equilibrium, bid constant value p flat on some non-trivial intervals of quantities.*

Proof. The proof resembles similar proofs in other auction contexts. Suppose agent i bids p on (q_ℓ^i, q_r^i) and bidder j bids p on (q_ℓ^j, q_r^j). Since the support of supply is $\left[0, \overline{Q}\right]$, it must be that $G^i(q_r^i; b^i) > G^i(q_\ell^i; b^i)$ and $G^j(q_r^j; b^j) > G^j(q_\ell^j; b^j)$. Let $\bar{q}^i = \mathbb{E}_Q[q^i | p(Q) = b(q_r^i)]$; without loss of generality, we may assume that agent i is such that $\bar{q}^i < q_r^i$. If $v^i(\bar{q}^i) < b^i(q_r^i)$, the agent has a profitable downward deviation. The agent also has a profitable deviation if $v^i(\bar{q}^i) \geq b^i(q_r^i)$: she can increase her bid slightly by $\lambda > 0$ on $[q_\ell^i, q_r^i)$ (enforcing monotonicity constraints as necessary to the left of q_ℓ^i), keeping her bid below value if necessary.[76] □

Lemma 9. *Bids are below values: $b^i(q) \leq v^i(q)$ for all relevant quantities, and $b^i(q) < v^i(q)$ for $q < \varphi^i(p(\overline{Q}))$.*

[76] Because we are conditioning on her expected quantity, we do not need to directly consider whether quantities are relevant.

Proof. Suppose that there exists q with $b^i(q) > v^i(q)$; because b^i is monotonic and v^i is continuous, there must exist a range (q_ℓ, q_r) of relevant quantities such that $b^i(q) > v^i(q)$ for all $q \in (q_\ell, q_r)$. The agent wins quantities from this range with positive probability, and hence the agent could profitably deviate to

$$\hat{b}^i(q) = \min\left\{b^i(q), v^i(q)\right\}.$$

Such a deviation never affects how she might be rationed, by the first part of this proof; hence it is necessarily utility-improving.

Now consider $q < \varphi^i\left(p\left(\overline{Q}\right)\right)$. If $b^i(q) = v^i(q)$ then monotonicity of b^i and Lipschitz-continuity of v^i imply that for small $\varepsilon > 0$ winning units $[q-\varepsilon, q]$ brings per unit profit lower than $M\varepsilon$, where M is the Lipschitz modulus of v. By lowering the bid for quantities $q' \in [q-\varepsilon, q+\varepsilon]$ to $\hat{b}^i(q') = \min\{v^i(q)-\varepsilon, b^i(q')\}$, the utility loss from losing the relevant quantities is at most $2M\varepsilon^2\left(G_i\left(q+\varepsilon; b^i\right) - G_i\left(q-\varepsilon; b^i\right)\right)$. Notice that the right-hand probability difference goes to zero as ε goes to zero. At the same time the cost savings from paying lower bids at quantities higher than $q+\varepsilon$ is (at least) of order ε^2. Hence this deviation is profitable, and it cannot be that $b^i(q) = v^i(q)$.

□

Lemma 10. *The market clearing price $p(Q)$ is strictly decreasing in supply Q, for all $Q \leq \overline{Q}^R$.*

Proof. We show first that the market clearing price is strictly decreasing in supply for all Q such that $p(Q) > \inf_{Q'} p(Q') = \underline{p}$. We then show that p is strictly decreasing at $\overline{Q}^R$ as long as for any bidder i residual supply $\sum_{j \neq i} \varphi^j(\cdot)$ has nonzero slope at $\underline{p}$. Since Corollary 6 shows that $b^i(\overline{q}^i) = \underline{p}$, Lemma 9 shows that bids are below values, and values are Lipschitz continuous, it follows that residual supply has nonzero slope at $\underline{p}$, and therefore the market clearing price is strictly decreasing in Q.

Since bids are weakly decreasing in quantity, the market price is weakly decreasing as a direct consequence of the market-clearing equation. If price is not weakly decreasing in quantity at some Q, then a small increase in Q will not only increase the price, but will weakly decrease the quantity allocated to each agent. This implies that total demand is no greater than Q, contradicting market clearing.

Lemma 8 is sufficient to imply that the market price must be strictly decreasing for all Q such that $p(Q) > \underline{p}$: at every price level at which at least two bidders pay with positive probability for some quantity, at most one of the submitted bid functions is flat. Furthermore, for no price level $p > \underline{p}$ that with positive probability a bidder pays for some quantity, we can have exactly one bidder, i, submitting a flat bid at price p on an interval of relevant quantities.[77] Indeed, in equilibrium bidder i cannot benefit by slightly reducing the bid on this entire interval; thus it must be that there is some other agent j whose bid function is right-continuous at price p. If $p = 0$, all opponents $j \neq i$ have a profitable deviation.[78] If $p > 0$, we appeal to Lemma 9. Given that i submits a flat bid and the bids of bidder j are strictly below her values for some non-trivial subset of quantities at which her bid is near p, bidder j can then profit by slightly raising her bid; this reasoning is similar to that given in the proof of Lemma 8.

We now show that $p(\cdot)$ is strictly decreasing for all Q. Otherwise, following Lemma 8, there is a bidder i who is submitting a flat bid at $\underline{p}$. Denote the left end of this bidder's flat by $\underline{q}_i = \inf\{q\colon b^i(q) = \underline{p}\}$; by assumption, $\underline{q}_i < \overline{q}_i$.[79] Let $\varepsilon, \lambda > 0$ and define a deviation

$$\hat{b}^{\varepsilon\lambda}(q) = \begin{cases} b^i(q) & \text{if } b^i(q) > \underline{p} + \lambda, \\ \underline{p} + \lambda & \text{if } b^i(q) \leq \underline{p} + \lambda \text{ and } q \leq \underline{q}_i + \varepsilon, \\ \underline{p} & \text{otherwise.} \end{cases}$$

That is, $\hat{b}^{\varepsilon\lambda}$ is b^i, with λ added for length ε at $\underline{q}_i$, and adjusting for the fact that bids must be monotone decreasing. Note that this deviation increases costs by at most $(\varepsilon + (\underline{q}_i - \varphi^i(\underline{p} + \lambda)))\lambda$, with at most probability one. When $q_i \in [\underline{q}_i, \underline{q}_i + \varepsilon]$, it increases the quantity allocation to (approximately) $\max\{\underline{q}_i + \varepsilon, q + \lambda M\}$, where M is the slope of residual supply at the minimum price, $M = \sum_{j \neq i} \varphi^j_p(\underline{p})$.[80] Let $\mu \equiv v^i(\underline{q}_i + \varepsilon) - (\underline{p} + \lambda)$; since bids are below values and values are strictly decreasing, $\mu > 0$ when ε and λ are sufficiently small. Then for the deviation to be nonoptimal, it must be that

$$\begin{aligned} \left(\varepsilon + \left(\underline{q}_i - \varphi^i\left(\underline{p} + \lambda\right)\right)\right)\lambda &\geq \mathbb{E}\left[\left(\max\left\{\varepsilon, q + \frac{\lambda}{M}\right\} - q\right)\mu \middle| q \in \left[\underline{q}_i, \underline{q}_i + \varepsilon\right]\right] \\ &= \mathbb{E}\left[\left(\max\left\{\varepsilon - q, \frac{\lambda}{M}\right\}\right)\mu \middle| q \in \left[\underline{q}_i, \underline{q}_i + \varepsilon\right]\right]. \end{aligned}$$

[77]We refer to any price level p that with positive probability a bidder pays for some quantity, as a relevant price level.

[78]Here we work in a model in which marginal utilities on all possible units is strictly positive. We could dispense with the strict positivity assumption by allowing negative bids.

[79]Because bidders are symmetric, it is not possible that $\overline{q}_i = 0$: in this case, bidder i almost surely receives 0 utility ex post, which is not optimal.

[80]Because we are ultimately letting ε and λ go to zero, this approximation is sufficient. Formally, we may consider $M' < M$ and allow δ to be small enough that the slope of residual supply never falls below M'.

Letting $\overline{Q}_{-i} = \sum_{j \neq i} \overline{q}_j$, this can be rewritten as

$$\left(\varepsilon + \left(\underline{q}_i - \varphi^i\left(\underline{p} + \lambda\right)\right)\right) \lambda \int_{\underline{q}_i}^{\underline{q}_i + \varepsilon} dF\left(q + \overline{Q}_{-i}\right) \geq \int_{\underline{q}_i}^{\underline{q}_i + \varepsilon} \max\left\{\varepsilon + \underline{q}_i - q, \frac{\lambda}{M}\right\} \mu dF\left(q + \overline{Q}_{-i}\right)$$
$$\geq \int_{\underline{q}_i}^{\underline{q}_i + \varepsilon - \frac{\lambda}{M}} \frac{\mu\lambda}{M} dF\left(q + \overline{Q}_{-i}\right).$$

The $\lambda > 0$ multipliers cancel; integrating through gives

$$\left(\varepsilon + \left(\underline{q}_i - \varphi^i\left(\underline{p} + \lambda\right)\right)\right)\left(F\left(\varepsilon + \overline{Q}_{-i}\right) - F\left(\overline{Q}_{-i}\right)\right) \geq \frac{\mu}{M}\left(F\left(\varepsilon - \frac{\lambda}{M} + \overline{Q}_{-i}\right) - F\left(\overline{Q}_{-i}\right)\right).$$

From here the argument is standard. For any $\varepsilon > 0$ there is $\lambda > 0$ such that $\varepsilon - \lambda/M \geq \varepsilon/2$ and $\underline{q}_i - \varphi^i(\underline{p} + \lambda) < \varepsilon/2$. Thus it must be that

$$\frac{3}{2}\varepsilon\left(F\left(\varepsilon + \overline{Q}_{-i}\right) - F\left(\overline{Q}_{-i}\right)\right) \geq \frac{\mu}{M}\left(F\left(\frac{1}{2}\varepsilon - \overline{Q}_{-i}\right) - F\left(\overline{Q}_{-i}\right)\right)$$
$$\iff \quad F\left(\varepsilon + \overline{Q}_{-i}\right) - F\left(\overline{Q}_{-i}\right) \geq \frac{\mu}{3M}\left[\frac{F\left(\frac{1}{2}\varepsilon - \overline{Q}_{-i}\right) - F\left(\overline{Q}_{-i}\right)}{\frac{1}{2}\varepsilon}\right].$$

This must hold for all $\varepsilon > 0$. Taking the limit as $\varepsilon \searrow 0$ gives

$$0 \geq \frac{\mu f\left(\overline{Q}_{-i}\right)}{3M}.$$

Since $f(\cdot) > 0$, this is a contradiction when$M$ is nonzero. In this case, bidder i has a profitable deviation. □

Corollary 7. *In any pure-strategy equilibrium, bid functions are strictly decreasing.*

We define the derivative of G^i with respect to b as follows. For any q and b^i, the mapping $t \mapsto G^i(q; b^i + t)$ is weakly decreasing in t, and hence differentiable almost everywhere. With some abuse of notation, whenever it exists we denote the derivative of this mapping with respect to t by $G^i_b(q; b^i)$.

Lemma 11. *For each agent i and almost every q we have:*

$$G^i_b\left(q; b^i\right) = f\left(q + \sum_{j \neq i} \varphi^j\left(b^i\left(q\right)\right)\right) \sum_{j \neq i} \varphi^j_p\left(b^i\left(q\right)\right).$$

Proof. By definition, $G^i(q;b^i) = \Pr(q^i \le q|b^i)$. From market clearing, this is

$$\begin{aligned} G^i\left(q;b^i\right) =& \Pr\left(Q \le q + \sum_{j\neq i}\varphi^j\left(b^i\left(q\right)\right)\right) \\ =& F\left(q + \sum_{j\neq i}\varphi^j\left(b^i\left(q\right)\right)\right). \end{aligned}$$

Where the demands φ^j of agents $j \neq i$ are differentiable, we have

$$G^i_b\left(q;b^i\right) = f\left(q + \sum_{j\neq i}\varphi^j\left(b^i\left(q\right)\right)\right)\sum_{j\neq i}\varphi^j_p\left(b^i\left(q\right)\right).$$

Since for all j, the demand function φ^j must be differentiable almost everywhere, the result follows. □

Lemma 12. *At points where $G^i_b\left(q;b^i\right)$ is well-defined, the first-order conditions of the pay-as-bid auction are given by*

$$-\left(v\left(q\right) - b^i\left(q\right)\right)G^i_b\left(q;b^i\right) = 1 - G^i\left(q;b^i\right).$$

In the case of pure strategies under symmetric bidder information,[81] *the first-order condition can be written as*

$$-\left(v\left(q\right) - b^i\left(q\right)\right)\left(\frac{d}{db}Q\left(b^i\left(q\right)\right) - \varphi^i_p\left(b^i\left(q\right)\right)\right) = H\left(Q\left(b^i\left(q\right)\right)\right),$$

where $Q\left(p\right)$ is the inverse of $p\left(Q\right)$.

Proof. The agent's maximization problem is given by

$$\max_b \int_0^{\overline{Q}}\int_0^q v\left(x\right) - b\left(x\right)dxdG^i\left(q;b\right).$$

Integrating by parts, we have

$$\max_b -\left[\left(1 - G^i\left(q;b\right)\right)\int_0^q v\left(x\right) - b\left(x\right)dx\right]|_{q=0}^{\overline{Q}} + \int_0^{\overline{Q}}\left(v\left(q\right) - b\left(q\right)\right)\left(1 - G^i\left(q;b\right)\right)dq.$$

[81] The definition of the derivative of bidder i's distribution of supply, G^i_b, obtained in Lemma 11, assumes pure strategies under symmetric bidder information. The first order condition derived here is invariant to the source of randomness in the bidder's allocation, but the statement in terms of aggregate demand holds only for pure strategies under symmetric bidder information.

In the first square bracket term, both multiplicands are bounded for $q \in [0, \overline{Q}]$, hence the fact that $1 - G^i(\overline{Q}; b) = 0$ for all b and $\int_0^0 v(x) - b(x)dx = 0$ for all b allows us to reduce the agent's optimization problem to

$$\max_b \int_0^{\overline{Q}} \left(v\left(q\right) - b\left(q\right)\right)\left(1 - G^i\left(q; b\right)\right) dq.$$

The calculus of variations gives us the necessary condition

$$-\left(1 - G^i\left(q; b^i\right)\right) - \left(v\left(q\right) - b^i\left(q\right)\right) G_b^i\left(q; b^i\right) = 0.$$

This holds at almost all points at which G_b^i is well-defined. Rearrangement yields the first expression for the first-order condition.

To derive the second expression, let us substitute into the above formula for G^i and G_b^i from the Lemma 11. We obtain

$$-\left(v\left(q\right) - b^i\left(q\right)\right) f\left(q + \sum_{j\neq i} \varphi^j\left(b^i\left(q\right)\right)\right)\left(\sum_{j\neq i} \varphi_p^j\left(b^i\left(q\right)\right)\right) = 1 - F\left(q + \sum_{j\neq i} \varphi^j\left(b^i\left(q\right)\right)\right),$$

Now, $Q\left(p\right)$ is well-defined since we have shown that p is strictly monotone. By Corollary 7 bids are strictly monotone in quantities and hence $q + \sum_{j\neq i} \varphi^j\left(b^i\left(q\right)\right) = Q\left(b^i\left(q\right)\right)$, and

$$-\left(v\left(q\right) - b^i\left(q\right)\right)\left(\sum_{j\neq i} \varphi_p^j\left(b^i\left(q\right)\right)\right) = H\left(Q\left(b^i\left(q\right)\right)\right).$$

Since $\sum_{j\neq i} \varphi_p^j\left(b^i\left(q\right)\right) = \frac{d}{db} Q\left(b^i\left(q\right)\right) - \varphi_p^i\left(b^i\left(q\right)\right)$, the second expression for the first order condition obtains. □

Lemma 13. *When bidders have symmetric information, equilibrium bidding strategies must be symmetric in all pure strategy equilibria:* $b^i = b$ *for all* i.

Proof. The proof proceeds by establishing an ordering of asymmetric bid functions. We use this ordering to show that equilibrium is symmetric in the $n = 2$ bidder case, and the result from the $n = 2$ bidder case provides tools for the general analysis. Intuitively, these results show that agents do not like receiving zero quantity when it is possible to receive a positive quantity; because this is a necessary feature of asymmetric putative equilibria, these bids are not best responses.

Note that for any agent i, $\sum_{j\neq i} \varphi_p^j(p) = Q_p(p) - \varphi_p^i(p)$. Then we can write the agent's first-order condition as

$$b^{i}\left(q\right)=v\left(q\right)+\left(\frac{1-F\left(Q\left(p\right)\right)}{f\left(Q\left(p\right)\right)}\right)\left(\frac{1}{Q_{p}\left(p\right)-\varphi_{p}^{i}\left(p\right)}\right).$$

Now suppose that two agents i,j have bid functions which differ on a set of positive measure; without loss, assume that $b^{i}>b^{j}$. Then there is a price p such that $\varphi^{i}(p)>\varphi^{j}(p)$, and $v(\varphi^{i}(p))<v(\varphi^{j}(p))$. Substituting into the agents' first-order conditions, this gives

$$\left(\frac{1-F\left(Q\left(p\right)\right)}{f\left(Q\left(p\right)\right)}\right)\left(\frac{1}{Q_{p}\left(p\right)-\varphi_{p}^{i}\left(p\right)}\right)>\left(\frac{1-F\left(Q\left(p\right)\right)}{f\left(Q\left(p\right)\right)}\right)\left(\frac{1}{Q_{p}\left(p\right)-\varphi_{p}^{j}\left(p\right)}\right).$$

Standard rearrangement gives

$$\varphi_{p}^{j}\left(p\right)<\varphi_{p}^{i}\left(p\right).$$

Thus whenever $\varphi^{i}(p)>\varphi^{j}(p)$, we have $\varphi_{p}^{i}(p)>\varphi_{p}^{j}(p)$. Recalling from Corollary 6 that bids must equal values at $q=\overline{Q}/n$, this implies that if there is any p such that $\varphi^{i}(p)>\varphi^{j}(p)$, then $\varphi^{i}>\varphi^{j}$.

Now consider the implications for the $n=2$ bidder case, and let $j\neq i$. Assume that there is p with $\varphi^{i}(p)>\varphi^{j}(p)>0$. Then there is some $\check{p}$ such that $\varphi^{j}(\check{p})=0$ and $\varphi^{i}(\check{p})>0$. Basic auction logic dictates that bidder i can never outbid the maximum bid of bidder j (i.e., it must be that $b^{i}(0)=b^{j}(0)$) thus it must be that bidder i's first-order condition does not apply for initial units, and she is submitting a flat bid. That is, $b^{i}(q)|_{q\leq\varphi^{i}(\check{p})}=\check{p}$. Now let $\varepsilon,\lambda>0$, and define a deviation $\hat{b}^{\varepsilon\lambda}$ for bidder 2,

$$\hat{b}^{\varepsilon\lambda}\left(q\right)=\begin{cases}b^{j}\left(0\right)+\lambda & \text{if } q\leq\varepsilon,\\ b^{j}\left(q\right) & \text{otherwise.}\end{cases}$$

Then for all $q\in(0,\varepsilon]$, $\hat{b}^{\varepsilon\lambda}(q)>b^{i}(q)$, and when the realized quantity is $Q\in(0,\varepsilon]$ bidder j wins the entire supply. To bound the additional utility, we see that for small $\varepsilon>0$ bidder j gains at least

$$\int_{0}^{\varepsilon}\left(v\left(x\right)-b^{j}\left(x\right)\right)dx\left(F\left(\varphi^{i}\left(\check{p}\right)\right)-F\left(\varepsilon\right)\right).$$

There is an extra cost paid as well; to bound this cost we will assume that it is paid with probability 1, and this cost is $(b^{j}(0)+\lambda)\varepsilon-\int_{0}^{\varepsilon}b^{j}(x)dx$. The deviation $\hat{b}^{\varepsilon\lambda}$ is profitable if the ratio of benefits to costs is greater than 1, hence we look at

$$\lim_{\lambda\searrow 0,\varepsilon\searrow 0}\frac{\int_0^\varepsilon\left(v\left(x\right)-b^j\left(x\right)\right)dx\left(F\left(\varphi^i\left(\check{p}\right)\right)-F\left(\varepsilon\right)\right)}{\left(b^j\left(0\right)+\lambda\right)\varepsilon-\int_0^\varepsilon b^j\left(x\right)dx}$$
$$=\lim_{\varepsilon\searrow 0}\frac{\int_0^\varepsilon\left(v\left(x\right)-b^j\left(x\right)\right)dx\left(F\left(\varphi^i\left(\check{p}\right)\right)-F\left(\varepsilon\right)\right)}{b^j\left(0\right)\varepsilon-\int_0^\varepsilon b^j\left(x\right)dx}.$$

The numerator and denominator both go to zero as $\varepsilon\searrow 0$; application of l'Hôpital's rule gives

$$=\lim_{\varepsilon\searrow 0}\frac{v\left(0\right)-b^j\left(0\right)}{0}=+\infty.$$

Then either the deviation to $\hat{b}^{\varepsilon\lambda}$ is profitable for bidder j (when $|b_q^j(0)|<\infty$), or bidder i may (essentially) costlessly reduce the initial flat of her bid function (when $|b_q^j(0)|=\infty$).[82]

Now consider the case of $n\geq 3$ agents. By the previous arguments we know that for small quantities submitted bid functions can be ranked (as can their inverses), and that at least two agents submit the highest possible bid function. Thus we focus attention on two selected bid functions,

$$\varphi^H\left(p\right)\equiv\max\left\{\varphi^i\left(p\right)\right\},$$
$$\varphi^L\left(p\right)\equiv\max\left\{\varphi^i\left(p\right):\varphi^i\left(p\right)<\varphi^H\left(p\right)\right\}.$$

Note that where submitted bid functions are symmetric φ^L will not be well-defined, but because we are attempting to prove that equilibrium bids are symmetric we need only pay attention to the asymmetric case. Lastly, let $m_H\equiv\#\{i\colon\varphi^i=\varphi^H\}$ and $m_L=\#\{i\colon\varphi^i=\varphi^L\}$ be the numbers of agents submitting each bid. As mentioned $m_H\geq 2$, and trivially $m_L\geq 1$; additionally, $m_H+m_L\leq n$. As before, there is $\check{p}$ such that $\varphi^L(\check{p})=0$, $\varphi^H(\check{p})>0$, and $\varphi^L(p)>0$ for all $p<\check{p}$. Corollary 7 shows that φ^H must be continuous, hence the equilibrium first order conditions imply

$$\lim_{p\searrow\check{p}}\left(m_H-1\right)\varphi_p^H\left(p\right)=\lim_{p\nearrow\check{p}}\left(m_H-1\right)\varphi_p^H\left(p\right)+m_L\varphi_p^L\left(p\right).$$

One obvious solution is $\lim_{p\nearrow\check{p}}\varphi_p^L(p)=0$; but since $\varphi_p^L\leq\varphi_p^H\leq 0$ this would imply that bids

[82] Implicit here is that $v(0)>b^j(0)=b^i(0)$, which follows from Lemma 9 but in this particular case is trivial: since bidder i is bidding flat to $\varphi^i(\check{p})$, if $v(0)=b^i(0)$ she is obtaining zero surplus on a positive measure of initial units. She would rather cut her bid and lose all of these units with some probability, saving payment for higher units and *gaining* probable gross utility.

are unboundedly negative, violating monotonicity constraints. Then we have

$$\lim_{p \searrow \check{p}} \varphi_p^H(p) = \lim_{p \nearrow \check{p}} \varphi_p^H(p) + \frac{m_L}{m_H - 1} \varphi_p^L(p) < 0.$$

Intuitively, the bid function b^H is steeper below $\varphi^H(\check{p})$ than above, and there is a kink at this point. This implies a discontinuity in a bidder L's first-order condition near $q = 0$. For p close to but less than $\check{p}$, the first-order condition is

$$\begin{aligned} & -\left(v\left(\varphi^L(p)\right) - p\right) f(Q(p)) \left(m_H \varphi_p^H(p) + (m_L - 1)\varphi_p^L(p)\right) - (1 - F(Q(p))) = 0, \\ \implies \quad & -\left(v\left(\varphi^L(p)\right) - p\right) f(Q(p)) \left((m_H - 1)\varphi_p^H(p) + m_L \varphi_p^L(p)\right) - (1 - F(Q(p))) > 0. \end{aligned}$$

Letting $p \nearrow \check{p}$, we know that the term $[(m_H - 1)\varphi_p^H(p) + m_L \varphi_p^L(p)]$ smoothly[83] approaches $\lim_{p \searrow \check{p}} (m_H - 1)\varphi_p^H(p)$, proportional to the marginal probability gained by a slight increase in bid from b^L near $\check{p}$ to $\tilde{b}^L > \check{p}$. Thus, the L bidder's second-order conditions are not satisfied near $q = 0$, and this is not an equilibrium. □

C Proofs for Section 3 (Pay-as-Bid Equilibrium)

For our proofs of Theorems 2, 3, and 4, we assume that the reserve price is $R = 0$. In this case, the maximum realizable quantity is $\overline{Q}^R = \overline{Q}$. In Supplementary Appendix C.5 we detail how these proofs must change to account for binding reserve prices.

C.1 Proof of Theorem 2 (Uniqueness)

Proof. From Lemma 12 and market clearing, we know that for all bidders

$$(p(Q) - v(q))\, G_b^i\left(q; b^i\right) = 1 - G^i\left(q; b^i\right).$$

Since Lemma 13 tells us that agents' strategies are symmetric, Lemma 11 allows us to write this as

$$\left(p(Q) - v\left(\frac{1}{n} Q\right)\right) (n-1)\, \varphi_p(p(Q)) = H(Q).$$

From market clearing, we know that $p(Q) = b(Q/n)$; hence $p_Q(Q) = b_q(Q/n)/n$. Additionally, standard rules of inverse functions give $\varphi_p(p(Q)) = 1/b_q(Q/n)$ almost everywhere. Thus

[83] Both φ^H and φ^L are continuous, hence $[(m_H - 1)\varphi_p^H + m_L \varphi_p^L]$ and $[m_H \varphi_p^H + (m_L - 1)\varphi_p^L]$ are continuous. This additionally implies that φ_p^L and φ_p^H are continuous.

we have

$$\left(p\left(Q\right)-v\left(\frac{1}{n}Q\right)\right)\frac{n-1}{n}=H\left(Q\right)p_{Q}\left(Q\right).$$

Now suppose that there are two solutions, p and $\hat{p}$. From Corollary 6 we know that $p(\overline{Q})=\hat{p}(\overline{Q})$. Suppose that there is a Q such that $\hat{p}(Q)>p(Q)$; taking Q near the supremum of Q for which this strict inequality obtains we conclude that $\hat{p}_Q(Q)<p_Q(Q)$.[84] But then we have

$$\hat{p}\left(Q\right)>p\left(Q\right)=v\left(\frac{1}{n}Q\right)+\left(\frac{n}{n-1}\right)H\left(Q\right)p_{Q}\left(Q\right)>v\left(\frac{1}{n}Q\right)+\left(\frac{n}{n-1}\right)H\left(Q\right)\hat{p}_{Q}\left(Q\right).$$

The presumed right-continuity of bids, and hence of p, allows us to conclude that if p solves the first-order conditions, $\hat{p}$ cannot. □

C.2 Proof of Theorem 3 (Bid Representation)

From the first order condition established in the proof of uniqueness, the equilibrium price satisfies

$$p_Q=p\tilde{H}-\hat{v}\tilde{H},$$

where $\hat{v}(x)=v(x/n)$, and $\tilde{H}(x)=[1/H(x)][(n-1)/n]$. The solution to this equation has general form

$$p\left(Q\right)=Ce^{\int_0^Q\tilde{H}(x)dx}-e^{\int_0^Q\tilde{H}(x)dx}\int_0^Q e^{-\int_0^x\tilde{H}(y)dy}\tilde{H}\left(x\right)\hat{v}\left(x\right)dx,$$

parametrized by $C\in\mathbb{R}$. Define $\rho=\frac{n-1}{n}\in[\frac{1}{2},1)$. We can see that $\tilde{H}=-\rho\frac{d}{dQ}\ln(1-F)$. Thus we have

$$e^{\int_0^t\tilde{H}(x)dx}=e^{-\rho\int_0^t\frac{d}{dx}\ln(1-F(x))dx}=e^{-\rho(\ln(1-F(t))-\ln 1)}=\left(1-F\left(t\right)\right)^{-\rho}.$$

Substituting and canceling, we have for $Q<\overline{Q}$:

$$p\left(Q\right)=\left(C-\rho\int_0^Q f\left(x\right)\left(1-F\left(x\right)\right)^{\rho-1}\hat{v}\left(x\right)dx\right)\left(1-F\left(Q\right)\right)^{-\rho}. \tag{5}$$

[84] The inequality inversion here from usual derivative-based approaches reflects the fact that we are "working backward" from $\overline{Q}$, while any solution must be weakly decreasing: thus a small *reduction* in Q should yield $\hat{p}(\overline{Q})=p(\overline{Q})\leq p<\hat{p}$.

Since $1 - F(\overline{Q}) = 0$, this implies that $C = \rho \int_0^Q f(x)(1 - F(x))^{\rho - 1} \hat{v}(x)\, dx$. The market clearing price is then given by

$$p(Q) = \rho \int_Q^{\overline{Q}} f(x)(1 - F(x))^{\rho-1} \hat{v}(x)\, dx \ (1 - F(Q))^{-\rho}.$$

Since $d/dy[F^{Q,n}(y)] = \rho f(y)(1-F(y))^{\rho-1}(1-F(Q))^{-\rho}$, our formula for market price obtains, and since we have proven earlier that the equilibrium bids are symmetric, the formula for bids obtains as well.

C.3 Proof of Theorem 4 (Existence) and an Alternative Form of the Existence Condition

Given an inverse bid function φ, define the local inverse hazard rate of residual supply $Y(q;b)$ by

$$Y(q;b) = \frac{1 - F(q + (n-1)\varphi(b))}{f(q + (n-1)\varphi(b))} = H(q + (n-1)\varphi(b)).$$

Y is the inverse hazard rate H evaluated at the total quantity demanded at a price of b if one agent demands q units and all others submit the (inverse) bid function φ.

The equilibrium existence condition in Theorem 4 can be weakened to the following: there exists a pure-strategy Bayesian Nash equilibrium whenever, for all $p \in (p(\overline{Q}), p(0))$ and all $Q < \overline{Q} - (n-1)\varphi(p)$,

$$E\left(\frac{1}{n}Q\right) = (n-1)\left(v\left(\frac{1}{n}Q\right) - p\right)\varphi_p(p) + Y\left(\frac{1}{n}Q; p\right) = 0$$
$$\implies E_q\left(\frac{1}{n}Q\right) = \frac{v_q\left(\frac{1}{n}Q\right) Y(\varphi(p); p)}{p - v(\varphi(p))} + Y_q\left(\frac{1}{n}Q; p\right) > 0.$$

The function E represents the equilibrium (negative) first-order conditions in the pay-as-bid auction; E_q is the cross-partial derivative of bidder utility with respect to bid and quantity.[85] Since, by Lemma 9, $v(\varphi(p)) - p > 0$ whenever $p > p(\overline{Q})$ the implication in Theorem 4 is equivalent to

$$Y(\varphi(p); p)\, v_q\left(\frac{1}{n}Q\right) - Y_q\left(\frac{1}{n}Q; p\right)(v(\varphi(p)) - p) < 0.$$

This resembles a standard second-order condition: the marginal gains to increasing the

[85] The cross-partial derivative in this context fills the role of a second derivative in a classical context. If whenever the first-order condition is satisfied—whenever $E(Q/n) = 0$—the derivative of the first-order condition with respect to its parameter (q) is strictly negative, there can be only one q at which the first-order condition is satisfied for any b. Then there is at most one b at which the first-order condition is satisfied for any q.

quantity demanded at a particular price are strictly decreasing.

Proof. We want to prove that the candidate equilibrium constructed in Theorem 3 is in fact an equilibrium. Let us this fix a bidder i whose incentives we will analyze, and assume that other bidders follow the strategies of Theorem 3 when bidding on quantities $q \leq \overline{Q}/n$ and that they bid $v(\overline{Q}/n)$ for quantities they never win.[86] Since bids and values are weakly decreasing, in equilibrium there is no incentive for bidder i to obtain any quantity $q > \overline{Q}/n$ and we only need to check that bidder i finds it optimal to submit bids prescribed by Theorem 3 for quantities $q < \overline{Q}/n$. Thus, agent i maximizes

$$\int_0^{\frac{\overline{Q}}{n}} \left(v\left(q\right) - b\left(q\right)\right)\left(1 - G\left(q; b\right)\right) dq$$

over weakly decreasing functions $b\left(\cdot\right)$.

We need to show that the maximizing function $b\left(\cdot\right)$ is given by Theorem 3, and because the bid function in Theorem 3 is strictly monotone, we can ignore the monotonicity constraint.[87] The problem can then be analyzed by pointwise maximization: for each quantity $q \in [0, \overline{Q}/n]$ the agent finds $b\left(q\right)$ that maximizes $\left(v\left(q\right) - b\left(q\right)\right)\left(1 - G\left(q; b\right)\right)$. Therefore, we can rely on one-dimensional optimization strategies to assert the sufficiency conditions for a maximum. As given in Lemma 12, the agent's first-order condition is

$$-\left(1 - G^i\left(q; b\right)\right) - \left(v\left(q\right) - b\right) G_b^i\left(q; b\right) = 0.$$

Recall that from any symmetric inverse bid of agent i's opponents, $G_b^i(q; b) = (n-1) f(q + (n-1)\varphi(b))\varphi_p(b)$. Then the first-order condition can be expressed as

$$(n-1)\left(v\left(q\right) - b\right)\varphi_p\left(b\right) + Y\left(q; b\right) = 0..$$

Suppose that there is $\hat{b}$ that also solves the first-order conditions for the bid for quantity q,[88]

$$(n-1)\left(v\left(q\right) - \hat{b}\right)\varphi_p\left(\hat{b}\right) + Y\left(q; \hat{b}\right) = 0.$$

Then since $b(\cdot)$ is continuous and any profitable deviation is such that $\hat{b} \in [b(\overline{Q}/n), b(0)]$

[86] When proving the analogue of Theorem 4 in the context of reserve prices, $\overline{Q}$ becomes $nv^{-1}(R)$, the aggregate quantity demanded at the reserve price R. The remainder of the argument does not change.

[87] In this regard, Theorem 4 is too strong: the conditions given are sufficient for no bid function—decreasing or otherwise—to generate more utility than the symmetric equilibrium given in Theorem 3.

[88] By the assumption of sufficient demand, bidding $\hat{b} = 0$ is never utility-improving. Further, bidding $\hat{b} > b(0)$ is also not utility-improving, so any solution to the first order conditions can be assumed to be internal.

there is some $\hat{q}$ such that $\hat{b} = b(\hat{q})$. At this point,

$$E\left(\hat{q};\hat{b}\right) \equiv (n-1)\left(v\left(\hat{q}\right)-\hat{b}\right)\varphi_p\left(\hat{b}\right)+Y\left(\hat{q};\hat{b}\right)=0.$$

If $\partial E/\partial q > 0$ (recall that E is the negative of the first-order condition) whenever $E(q;b) = 0$ then $E(\cdot;b)$ has a unique zero (if it has any). Then there is at most one solution to the first-order conditions; since the bid representation formula in Theorem 3 gives a closed-form solution for bids and the first-order conditions have a unique solution, the bids given in the representation theorem are an equilibrium. Calculation gives

$$\frac{\partial E}{\partial q}=(n-1)\,v_q\left(q\right)\varphi_p\left(b\right)+Y_q\left(q;b\right)>0.$$

In the symmetric solution to the market clearing equation we have already seen that $(n-1)\varphi_p(b) = Y(\varphi(b))/(b - v(\varphi(b)))$. Substituting this in gives the desired result. □

C.4 Verification of an Existence Example

Linear marginal values with generalized Pareto distribution of supply. For generalized Pareto distributions with parameter $\alpha > 0$,

$$\begin{aligned} 1-F\left(x\right)&=\left(1-\frac{x}{\overline{Q}}\right)^{\alpha}, & f\left(x\right)&=\frac{\alpha}{\overline{Q}}\left(1-\frac{x}{\overline{Q}}\right)^{\alpha-1}; \\ H\left(x\right)&=\frac{1}{\alpha}\left(\overline{Q}-x\right), & H_q\left(x\right)&=-\frac{1}{\alpha}. \end{aligned}$$

Then with linear market values $v(q) = \beta_0 - q\beta_q$,

$$-\frac{1}{\alpha}\left(\overline{Q}-n\varphi\left(p\right)\right)\beta_q+\frac{1}{\alpha}\left(\beta_0-\varphi\left(p\right)\beta_q-p\right)\propto\beta_0-\left(\overline{Q}-(n-1)\,\varphi\left(p\right)\right)\beta_q-p.$$

For all $Q < \overline{Q}$, $p(Q) > p(\overline{Q})$ and $\overline{Q} > n\varphi(p(Q))$; hence for all $Q < \overline{Q}$,

$$\beta_0-\left(\overline{Q}-(n-1)\,\varphi\left(p\right)\right)\beta_q-p<\beta_0-\frac{1}{n}\overline{Q}\beta_q-p\left(\overline{Q}\right)=0.$$

Then the existence condition is satisfied for all $Q \in [0, \overline{Q})$.

C.5 Modifying the Proofs to Allow for Reserve Prices

The bound on market price established in Theorem 1 implies that a binding reserve price is equivalent to creating an atom in the supply distribution at the quantity at which marginal value equals the reserve price. In order to extend the previous proofs to the setting that allows reserve prices (as the results are stated in the main text), we therefore need to extend them to distributions in which there might be an atom at the upper bound of support $\overline{Q}$.[89] All our results remain true, and the proofs go through without much change except for the end of the proof of Theorem 3, where more care is needed.

The proof of Theorem 3 goes through until the claim that $1-F(\overline{Q})=0$; in the presence of an atom at $\overline{Q}$ this claim is no longer valid. We thus proceed as follows. We multiply both sides of equation (5) by $(1-F(Q))^{\rho}$ and conclude that

$$p(Q)(1-F(Q))^{\rho}=C-\rho\int_0^Q f(x)(1-F(x))^{\rho-1}\hat{v}(x)\,dx.$$

Now, let $\vec{F}(\overline{Q})\equiv\lim_{Q'\nearrow\overline{Q}}F(Q')$. Because the market price and the right-hand integral are continuous as $Q\nearrow\overline{Q}$, we have

$$p\left(\overline{Q}\right)\left(1-\vec{F}\left(\overline{Q}\right)\right)=C-\rho\int_0^{\overline{Q}} f(x)(1-F(x))^{\rho-1}\hat{v}(x)\,dx.$$

The parameter C is determined by this equation. The market price function is then

$$p(Q)=\left(\frac{1-\vec{F}\left(\overline{Q}\right)}{1-F(Q)}\right)^{\rho}p\left(\overline{Q}\right)+\rho\int_Q^{\overline{Q}} f(x)(1-F(x))^{\rho-1}\hat{v}(x)\,dx\ \ (1-F(Q))^{-\rho}. \qquad (6)$$

Recall from Corollary 6 that $p(\overline{Q})=v(\overline{Q}/n)$. Extending our notation to the auxiliary distribution $F^{Q,n}$, we also have

$$F^{Q,n}(\overline{Q})-\vec{F}^{Q,n}(\overline{Q})=1-\vec{F}^{Q,n}(\overline{Q})=\left(\frac{1-\vec{F}\left(\overline{Q}\right)}{1-F(Q)}\right)^{\rho}.$$

Since $d/dy[F^{Q,n}(y)]=\rho f(y)(1-F(y))^{\rho-1}(1-F(Q))^{-\rho}$ for all $Q,y<\overline{Q}$, we have

$$\begin{aligned}
p(Q) &= \left(F^{Q,n}(\overline{Q})-\vec{F}^{Q,n}(\overline{Q})\right)\hat{v}\left(\overline{Q}\right)+\int_Q^{\overline{Q}}\hat{v}(x)\frac{d}{dy}\left[F^{Q,n}(y)\right]_{y=x}dx\\
&= \int_Q^{\overline{Q}}\hat{v}(x)\,dF^{Q,n}(x),
\end{aligned}$$

[89] Starting with a given supply distribution F with support $[0,\overline{Q}]$ and moving all probability from $[\overline{Q}^R,\overline{Q}]$ to an atom at $\overline{Q}^R$ results in a new distribution $\tilde{F}$ with support $[0,\overline{Q}^R]$, with an atom at $\overline{Q}^R$. All results apply to this new distribution, thus it is without loss of generality to assume that the mass point is at $\overline{Q}$.

proving our formula for equilibrium stop-out price in the presence of an atom at $\overline{Q}$. Noting that $\overline{Q}^R < \overline{Q}$ implies an atom in the realized allocation distribution at $\overline{Q}^R$, equation 2 in Theorem 3 follows. Since equilibrium is symmetric, equation 1 is an immediate corollary. □

D Proofs for Section 4 (Designing Pay-as-Bid Auctions): Proof of Theorem 5

Theorem 5 shows that, when the designer is constrained to a reserve price R and a distribution over supply F, the optimal mechanism is deterministic. This is distinct and does not follow from the analysis in Appendix A, which shows that (under regularity conditions on demand) a seller who can implement stochastic elastic supply prefers to implement a deterministic elastic supply curve. In general, fixed supply $Q^\star$ and reserve $R^\star$ is insufficiently elastic to obtain monopoly rents from all bidder signals s, and a seller who can implement an elastic supply curve will strictly prefer to do so.

Proof of Theorem 5. Consider a pure-strategy equilibrium in a pay-as-bid auction with reserve price R and supply distribution F. In Section 3 we proved that the equilibrium is essentially unique and symmetric. Furthermore, in equilibrium, for any relevant quantity q, each bidder's bid equals the resulting market-clearing price when quantity $Q = nq$ is sold; we denote this market clearing price $p(Q; R, s)$, suppressing in the notation the price's dependence on F as it is constant. We denote the resulting equilibrium revenue by $\pi(Q; R, s)$ and we write $\hat{v}(y; s) = v(y/n; s)$ for a bidder's marginal value from his or her share of quantity sold y.

Proof. The seller maximizes the expected revenue $\mathbb{E}\left[\pi^F\right] = \mathbb{E}_s \int_0^{\overline{Q}} \pi\left(Q; R, s\right) dF\left(Q\right)$, where π^F denotes the seller's profits when bidders bid against distribution of supply F. When bidders' values are low relative to the reserve price, and the realized quantity is high, the reserve price is binding and the bidders receive only a partial allocation. Expected revenue is

$$\mathbb{E}\left[\pi^F\right] = \mathbb{E}_s \int_0^{\overline{Q}} \int_0^{Q^R(y,s)} p\left(x; R, s\right) dx dF\left(y\right). \tag{7}$$

Integrating by parts gives

$$\begin{aligned}\mathbb{E}\left[\pi^F\right] =& \mathbb{E}_s \Bigg\{ \left[-\left(1 - F\left(y\right)\right) \int_0^{Q^R(y,s)} p\left(x; R, s\right) dx \right] \Bigg|_{y=0}^{\overline{Q}} \\ &+ \int_0^{\overline{Q}} \left(1 - F\left(y\right)\right) p\left(Q^R\left(y, s\right); s\right) dQ^R\left(y, s\right) \Bigg\},\end{aligned}$$

where the first addend is zero. Recognizing that Q is continuous in y and that $Q_y^R(y,s) = 1$ for $v(y/n;s) > R$ and $Q_y^R(y,s) = 0$ for $v(y/n;s) < R$, we can thus express the expected revenue as

$$\mathbb{E}\left[\pi^F\right] = \mathbb{E}_s \int_0^{\overline{Q}^R(s)} \left(1 - F\left(y\right)\right) p\left(Q^R\left(y,s\right);s\right) dy.$$

Applying our Theorem 3 gives

$$\mathbb{E}\left[\pi^F\right] = \mathbb{E}_s \int_0^{\overline{Q}^R(s)} \left(1 - F\left(y\right)\right) \left[\left(1 - F^{y,n}\left(\overline{Q}^R\left(s\right)\right)\right) \hat{v}\left(\overline{Q}^R\left(s\right);s\right) + \int_y^{\overline{Q}^R(s)} \hat{v}\left(x;s\right) dF^{y,n}\left(x\right)\right] dy, \tag{8}$$

where $F^{y,n}\left(x\right) = 1 - \left(\frac{1-F(x)}{1-F(y)}\right)^{\frac{n-1}{n}}$ is the c.d.f. of the weighting distribution from the theorem.[90] $\square$

Applying integration by parts to the inner integral and substituting in for $F^{y,n}$ gives

$$\mathbb{E}\left[\pi^F\right] = \mathbb{E}_s \int_0^{\overline{Q}^R(s)} \left(1 - F\left(y\right)\right) \hat{v}\left(y;s\right) + \left(1 - F\left(y\right)\right)^{\frac{1}{n}} \int_y^{\overline{Q}^R(s)} \hat{v}_q\left(x;s\right) \left(1 - F\left(x\right)\right)^{\frac{n-1}{n}} dxdy. \tag{9}$$

We may change the order of integration of the right-hand double integral to obtain

$$\begin{aligned}\int_0^{\overline{Q}^R(s)} \left(1 - F\left(y\right)\right)^{\frac{1}{n}} \int_y^{\overline{Q}^R(s)} \hat{v}_q\left(x;s\right) \left(1 - F\left(x\right)\right)^{\frac{n-1}{n}} dxdy &= \int_0^{\overline{Q}^R(s)} \int_0^x \left(1 - F\left(y\right)\right)^{\frac{1}{n}} dy \hat{v}_q\left(x;s\right) \left(1 - F\left(x\right)\right)^{\frac{n-1}{n}} \\ &\leq \int_0^{\overline{Q}^R(s)} x\hat{v}_q\left(x;s\right) \left(1 - F\left(x\right)\right) dx,\end{aligned}$$

where the inequality follows from the facts that $\hat{v}_q \leq 0$, and $1 - F(y) \geq 1 - F(x)$ for $y \leq x$. Substituting y for x and plugging this bound in the above expression for expected profits, we have

$$\mathbb{E}\left[\pi^F\right] \leq \mathbb{E}_s \int_0^{\overline{Q}^R(s)} \left(1 - F\left(y\right)\right) \left(\hat{v}\left(y;s\right) + y\hat{v}_q\left(y;s\right)\right) dy.$$

Notice that $x\hat{v}_q(x;s) + \hat{v}(x;s) = \pi_q^\delta(x;s)$, where $\pi^\delta(x;s) = x\hat{v}(x;s)$ is the revenue from selling

[90] The outer integral in equation (8) is bounded to $[0, Q^*(s)]$, thus $y \leq Q^*(s)$ for all y and $F^{y,n}(Q^*(s))$ is well-defined. The left-hand addend in the integral results from the fact that, when $Q^*(s) < \overline{Q}$—that is, when signal-s bidders have low values for the maximum quantity, $\hat{v}(\overline{Q};s) < R$—there is a mass point in the resulting distribution of realized aggregate allocation at $Q^*(s)$; this same expression is seen in equation (6) in Appendix (C.5).

quantity x at price $\hat{v}(x;s)$. Integrating by parts gives

$$
\begin{aligned}
\mathbb{E}\left[\pi^F\right] &\leq \mathbb{E}_s\left[\int_0^{\overline{Q}^R(s)} \pi_q^\delta\left(x;s\right)\left(1-F\left(x\right)\right)dx\right] \\
&= \mathbb{E}_s\left[\pi^\delta\left(\overline{Q}^R\left(s\right);s\right)\left(1-F\left(\overline{Q}^R\left(s\right)\right)\right)+\int_0^{\overline{Q}^R(s)} \pi^\delta\left(x;s\right)dF\left(x\right)\right] \\
&= \mathbb{E}_s\left[\int_0^{\overline{Q}} \pi^\delta\left(Q^R\left(x,s\right);s\right)dF\left(x\right)\right]. \qquad (10)
\end{aligned}
$$

Thus,

$$
\mathbb{E}\left[\pi^F\right] \leq \int_0^{\overline{Q}} \mathbb{E}_s\left[\pi^\delta\left(Q^R\left(x,s\right);s\right)\right]dF\left(x\right).
$$

Since there are no cross-terms in this integral, the right-hand side is maximized at a degenerate distribution which maximizes $\mathbb{E}_s[\pi^\delta(Q^R(x,s);s)]$. But this is exactly the problem of choosing optimal feasible deterministic supply given the reserve price R. It follows that expected revenue is weakly dominated by expected revenue with optimal deterministic supply, hence optimal supply is deterministic. □

Remark 4. The proof of Theorem 5 remains valid for the profit maximization problem of a seller facing increasing marginal costs. Let $C(Q)$ be the seller's cost of supplying quantity Q, and assume that $c(Q) = dC(Q)/dQ$ is positive and weakly increasing. Equation (7) for expected profits in the proof of Theorem 5 must be adjusted to

$$
\mathbb{E}\left[\pi^F\right] = \mathbb{E}_s\int_0^{\overline{Q}}\int_0^{Q^R(y,s)} p\left(x;R,s\right)-c\left(x\right)dxdF\left(y\right).
$$

Subsequent integration by parts remains valid, and equation (9) becomes

$$
\mathbb{E}\left[\pi^F\right] = \mathbb{E}_s\int_0^{\overline{Q}^R(s)}\left(1-F\left(y\right)\right)\left(\hat{v}\left(y;s\right)-c\left(y\right)\right)+\left(1-F\left(y\right)\right)^{\frac{1}{n}}\int_y^{\overline{Q}^R(s)}\hat{v}_q\left(x;s\right)\left(1-F\left(x\right)\right)^{\frac{n-1}{n}}dxdy.
$$

As before, letting $\pi^\delta(q;s,c)$ be monopoly profits when quantity q is sold to type s given marginal cost curve c, we obtain

$$
\mathbb{E}\left[\pi^F\right] \leq \mathbb{E}_s\left[\int_0^{\overline{Q}} \pi^\delta\left(Q^R\left(x,s\right);s,c\right)dF\left(x\right)\right].
$$

The remainder of the proof is immediate.

E Proofs for Section 5 (The Auction Design Game): Proof of Theorem 7

In the proof below we decorate market outcome functions with superscripts denoting the relevant mechanism, where helpful. For example, $p^{\star\mathrm{UP}}$ is the market-clearing price in the uniform-price auction and $p^{\star\mathrm{PAB}}$ is the market-clearing price in the pay-as-bid auction.

Proof of Theorem 7. As discussed in Theorem 5 and Lemma 1, we may restrict attention to optimal deterministic supply distributions in both the pay-as-bid and uniform-price auctions. Revenue maximization may then be expressed as a per-agent quantity $q^\star$ and market price $p^\star$; for signals s such that $v(q^\star; s) \geq p^\star$ it is without loss to assume that the total allocation is $nq^\star$—there is sufficient demand for the total quantity at the reserve price—while for signals s such that $v(q^\star; s) < p^\star$ it is clear that some total quantity $nq' < nq^\star$ will be allocated. The seller's expected revenue is then an expectation over bidder signals,

$$\mathbb{E}_s[\pi] = \mathbb{E}_s\left[nq\left(q^\star, p^\star; s\right) \cdot p\left(q^\star, p^\star; s\right)\right].$$

The quantity allocated under the uniform-price auction equals the quantity allocated under the pay-as-bid auction, $q^{\mathrm{UP}}(q^\star, p^\star; s) = q^{\mathrm{PAB}}(q^\star, p^\star; s)$, whenever $v(\cdot; s)$ is strictly decreasing at this quantity, or when $v(\cdot; s) > p^\star$ at this quantity.[91] Since we have assumed that $v(\cdot; s)$ is strictly decreasing, the quantity allocation depends only on $q^\star$ and $p^\star$ and not on the mechanism employed. Additionally, it is the case that $p^{\star\mathrm{UP}}(q^\star, p^\star; s) = p^{\star\mathrm{PAB}}(q^\star, p^\star; s)$ whenever $v(q^\star; s) < p^\star$. Let $\underline{\mathcal{S}}$ be the set of such s,

$$\underline{\mathcal{S}} = \underline{S}\left(nq^\star, p^\star\right) = \left\{s' \colon v\left(q^\star; s\right) < p^\star\right\}.$$

Then we have

$$\mathbb{E}_s[\pi] = p^\star \Pr\left(s \in \underline{\mathcal{S}}\right) \mathbb{E}_s\left[nq\left(q^\star, p^\star; s\right) | s \in \underline{\mathcal{S}}\right] + nq^\star \Pr\left(s \notin \underline{\mathcal{S}}\right) \mathbb{E}_s\left[p\left(q^\star, p^\star; s\right) | s \notin \underline{\mathcal{S}}\right].$$

The left-hand term is independent of the mechanism employed, while the right-hand term depends on the mechanism only via the expected market-clearing price. In the pay-as-bid auction, we have seen that $p(q^\star, p^\star; s) = v(q^\star; s)$ for all $s \notin \underline{\mathcal{S}}$, while in the uniform-price auction any price $p \in [p^\star, v(q^\star; s)]$ is supportable in equilibrium. It follows that the

[91] In the latter case there is excess demand, so all units will be allocated. In the former case all units are allocated at the reserve price; there is a possible difference in allocation when bidders' marginal values are flat over an interval of quantities at the reserve price, since bidders are indifferent between receiving and not receiving these quantities.

pay-as-bid auction weakly revenue dominates the uniform-price auction, and generally will strictly do so. That the seller-optimal equilibrium of the uniform-price auction is revenue-equivalent to the unique equilibrium of the pay-as-bid auction arises from the selection of $p^{\star\mathrm{UP}}(q^\star, p^\star; s) = v(q^\star; s)$ for all $s \notin \underline{\mathcal{S}}$. □

F Proofs for Section 6 (Asymmetric Information among Bidders): Proof of Theorem 12

Proof. The analogue of Lemma 3 obtains for (not necessarily optimal) supply and reserve as long as they are deterministic; the proof follows the same steps. The first statement follows then from this deterministic analogue of Lemma 3 and from Lemma 4. To prove the second statement, consider a uniform-price auction where bids, conditional on common signal s, are bounded below by $\underline{b}(s) = \max\{R, \operatorname{ess\,inf}_{\zeta|s} v(Q/n; \zeta)\}$: bidding below $\underline{b}(s)$ cannot yield additional quantity, and by construction, when $b \geq \underline{b}(s)$ the marginal value for all units obtained is weakly positive. It follows that there is an equilibrium in which bids are at least $\underline{b}(s)$, and the second claim follows. □

G Proofs for Appendix A (Elastic Supply)

G.1 Proof of Theorem 14 (Uniqueness with Elastic Supply)

Proof. The analysis from the proof of Theorem 1 allows us to conclude that on the maximum unit each bidder might receive, the bidder pays her marginal value. Letting $\hat{Q}(s)$ be the aggregate quantity awarded in equilibrium under supply curve $Q^\star(s)$, it cannot be that $p^\star(\hat{Q}(s); s) > \hat{v}(\hat{Q}(s); s)$, since bids are below values. If, instead, $p^\star(\hat{Q}(s); s) < \hat{v}(\hat{Q}(s); s)$, the arguments from the proof of Theorem 1 apply; indeed, they are strengthened by the fact that a small increase in bid increases allocation not only by beating opponent bids, but also by increasing the market price and moving up the supply curve.

Because each bidder bids $b^\star(\hat{Q}(s)/n; s) = v(\hat{Q}(s)/n; s)$ in any equilibrium, each bidder's allocation is $\hat{Q}(s)/n$. This allocation is deterministic, conditional on the bidder-common signal s. Then any bid curve b such that $b(q) > v(\hat{Q}(s)/n; s)$ for some $q > 0$ is wasteful: it does not affect the resulting allocation, and $\int_0^{\hat{Q}(s)/n} b(q)dq > \int_0^{\hat{Q}(s)/n} b^\star(q; s)dq$. It follows that $b^\star(q) = v(\hat{Q}(s)/n; s)$ for all $q \leq \hat{Q}(s)/n$, and equilibrium bids are unique for all relevant quantities. □

G.2 Proof of Lemma 5

As we consider the special case of the seller who knows the bidders' values, we simplify notation and suppress the signal while writing value and bid functions.

G.2.1 Preliminary Definitions

Recall that we defined the supply reserve distribution $K(Q;R)$ in Appendix A. For simplicity, we carry out the analysis under the assumption that supply-reserve distribution K is continuously differentiable. We show that this assumption may be dropped in Remark 5.

Holding the supply-reserve distribution K fixed, fix a bidder i and consider the aggregate demand of her opponents. Allowing for mixed strategies and asymmetric and asymmetrically-informed bidders, we denote the aggregate demand of bidder i's opponents by $Q(\cdot;\xi)$, where ξ indexes the joint distribution of her opponents' potentially mixed strategies. As with supply-reserve distribution K, we assume that aggregate demand Q is continuously differentiable, and drop this assumption in Remark 5. Although we separately specify the supply-reserve distribution K and the mixed strategy index ξ because the former is controlled by the seller while the latter is not, the set of bidder's best responses does not depend on the source of randomness in a bidder's residual supply. Bidders' ex post utility is determined by realized quantity and payment, and thus the dependence of interim utility on the joint distribution of quantity and payment is unaffected by the introduction of a random reserve price, asymmetry and asymmetric information among bidders, and the possibility of mixed strategies. Thus, the bidder's first order condition is unchanged from the analysis in Lemma 12 (in Supplementary Appendix B), and random reserve affects only the distribution of realized quantity. In the language of Lemma 12,

$$\begin{aligned} G^i(q;b) &= \mathbb{E}_\xi\left[K\left(q+Q\left(b;\xi\right);b\right)\right], \\ \text{and } G^i_b(q;b) &= \mathbb{E}_\xi\left[K_Q\left(q+Q\left(b;\xi\right);b\right)Q_p\left(b;\xi\right)+K_R\left(q+Q\left(b;\xi\right);b\right)\right]. \end{aligned} \tag{11}$$

E.g. when the reserve price is fixed, $K_R = 0$ for all relevant prices, and (11) is identical to what we find in equation (11).

We aim to show that the seller can induce the same bidder behavior by implementing a random reserve without constraining supply, in which case $K_Q = 0$, and the bidder's pointwise first order condition is

$$\left(v\left(q\right)-b\left(q\right)\right)\mathbb{E}_\xi\left[K_R\left(q+Q\left(b\left(q\right);\xi\right);b\right)\right]=\mathbb{E}_\xi\left[K\left(q+Q\left(b\left(q\right);\xi\right);b\right)\right].$$

As $K_Q = 0$ implies that K is independent of q (and thus Q is independent of ξ), we write this in terms of only the distribution of reserve prices F^R,

$$(v(\varphi(p)) - p) F_p^R(p) = F^R(p).$$

Thus a key simplification associated with random reserve and unconstrained supply is that the optimal bid is determined by the reserve distribution F^R and does not depend on opponent bids. Furthermore, for each quantity the optimal bid is either pointwise optimal, or this quantity is part of an interval on which the first order conditions are ironed, cf. Woodward [2016]. We capture these optimality conditions in the concept of first-order optimal bids defined as follow.

Definition 3. Given a distribution of reserve prices F^R, we say that b is *first-order optimal* with respect to F^R if:

1. If b is strictly decreasing at q, then it solves the pointwise first order condition: $(v(q) - b(q))F_p^R(b(q)) = F^R(b(q))$.
2. If b is constant in a neighborhood of q then $b(q)$ is a mass point of F^R and it solves the ironed first order condition:

$$\left(F^R(b(q)) - F^R(\underline{b})\right)(v(\overline{\varphi}(p)) - \underline{b}) = (b(q) - \underline{b}) F^R\left(\underline{p}\right), \text{ where } \underline{b} = \lim_{q' \searrow \overline{\varphi}(p)} b(q').$$

Intuitively, the ironing conditions state that the marginal gain from slightly extending the constant interval (marginal additional quantity with probability $F^R(b(q)) - F^R(\underline{b})$) must equal the marginal cost from the same (marginal additional payment with probability $F^R(\underline{b})$). As b is weakly decreasing, any quantity q belongs to either an interval on which b is flat or to an interval on which b is strictly decreasing (and it might be an endpoint of both types of intervals simultaneously). The structure of these intervals can be complex, but there is at most a countable number of them.

Although optimal bids are first-order optimal the converse need not be true: first-order optimality only implies that a bid satisfies pointwise first order conditions where applicable, and ironing conditions elsewhere. In deriving the revenue bounds below, we assume only that the first-order conditions are satisfied, not that bids are optimal. Because any (globally) optimal bid function satisfies the first-order optimality conditions above, the bound on revenues applies to optimal bids.

Let $G^K(\cdot; b, Q)$ be the distribution of realized quantity given stochastic-elastic supply K, bid function b, and, potentially random, residual supply Q, and let $G^R(\cdot; b)$ be the distribution of realized quantity given reserve distribution F^R and bid function b. As mentioned above,

G^R does not depend on Q because, under random reserve, supply does not depend on opponent bids. Letting ξ represent randomness in residual supply (e.g., mixed strategies for a bidder's opponents)[92] we have

$$
\begin{aligned}
G^R(q;b) &= 1 - F^R(b(q)),\\
\frac{d}{dq}G^R(q;b) &= -F_p^R(b(q))\, b_q(q);\\
G^K(q;b,Q) &= \mathbb{E}_\xi\left[K\left(q + Q\left(b(q);\xi\right), b(q)\right)\right],\\
\frac{d}{db}G^K(q;b,Q) &= \mathbb{E}_\xi\left[K_q\left(q + Q\left(b(q);\xi\right)\right) Q_p\left(b(q);\xi\right) + K_p\left(q + Q\left(b(q);\xi\right), b(q)\right)\right],\\
\frac{d}{dq}G^K(q;b,Q) &= \frac{d}{db}G^K(q;b,Q)\, b_q(q) + \mathbb{E}_\xi\left[K_q\left(q + Q\left(b(q);\xi\right)\right)\right]. \qquad (12)
\end{aligned}
$$

The expected revenue from bidder i when the bidder bids b and the bid leads to quantity distribution $G^{\cdot}$ is given by $\pi(b;G^{\cdot}) = \int_0^Q \int_0^q b(x)\,dx\,dG^{\cdot}(q)$.

G.2.2 The Optimality of Random Reserve with Known Values

We begin with a bid function b which is a best response to residual supply distribution $G^i(\cdot;b)$ and construct a reserve price distribution and bidder's best response to this new distribution that raise more revenue.

Lemma 14. *Let b be a best response bid curve under residual supply distribution G^i, generated by supply-reserve distribution K and stochastic aggregate demand Q. There is a reserve distribution F^R and a first order best response b^R to F^R such that $\pi\left(b^R;G^R\right) \geq \pi\left(b;G^i\right)$.*

While the bound on revenue in Lemma 14 might depend on the equilibrium selected, the subsequent analysis will show that this bound is weakly lower than the revenue in a unique equilibrium under deterministic elastic supply.

Proof. For clarity, we proceed under the assumption that supply-reserve distribution studied K and aggregate residual demand Q are continuously differentiable. Following the derivation of the result for smooth K and Q, we comment on extending the argument to potentially discontinuous K and Q.

Any distribution of reserve F^R and the first-order optimal response b^R induce a distribution (c.d.f.) $\tilde{G}^R$ of quantities sold to bidder i. As a step in constructing b^R and F^R, we first construct an auxiliary distribution G^R which is not necessarily equal to $\tilde{G}^R$. As a

[92] In the main text we focus on pure strategies. In this analysis we allow for mixed strategies, allowing us to show that all randomness—exogenous or otherwise—is detrimental to the seller's revenue.

preparatory step to construct the latter distribution, recall that the discussion of the previous subsection shows that under a random reserve price that induces differentiable quantity distribution G^R, in any interval in which b is strictly decreasing. We will define G^R so that the pointwise first order conditions of an agent bidding b are satisfied; that is,

$$-\left(v\left(q\right)-b\left(q\right)\right)G_q^R\left(q\right)=\left(1-G^R\left(q\right)\right)b_q\left(q\right),$$

and thus

$$\frac{d}{dq}\ln\left[1-G^R\left(q\right)\right]=\frac{b_q\left(q\right)}{v\left(q\right)-b\left(q\right)}.$$

Given any initial value of $G^R(q)$ (initial condition of the ODE), we can solve this differential equation for any differentiable $b<v\left(q\right)$. In particular, for any quantity $\tilde{q}$ such that b is strictly decreasing on $(\tilde{q},q)$, we obtain

$$G^R\left(\tilde{q}\right)=1-\exp\left(\int_q^{\tilde{q}}\frac{b_q\left(x\right)}{v\left(x\right)-b\left(x\right)}dx\right)\left[1-G^R\left(q\right)\right]. \tag{13}$$

We now construct G^R and we show that $G^R\succeq_{\text{FOSD}}G^K$; in particular, G^R puts more weight on larger quantities than G^K does. To start, let $G^R(0)=G^K(0)$. At the left endpoint of any maximal interval $(\tilde{q}_\ell,\tilde{q}_r)$ on which b is strictly decreasing, we define G^R so that $G^R(\tilde{q}_\ell)=G^K(\tilde{q}_\ell)$, and we define G^R on the interior of $(\tilde{q}_\ell,\tilde{q}_r)$ so that b satisfies the first-order ODE given the initial condition $G^R(\tilde{q}_\ell)$.[93] In particular, the first-order ODE determines the value at the right endpoint of the strictly decreasing b interval, $G^R(\tilde{q}_r)$. For any maximal open interval (q_ℓ,q_r) on which b is constant, let the value at the right endpoint be $G^R(q_r)=G^K(q_r)$.[94] Notice that for any maximal interval (q_ℓ,q_r) on which b is constant, q_ℓ is either 0 or equal to a limit of a sequence of the right end points of maximal intervals.[95] We will see below that the values of G^R on this sequence are monotonic. Since they are also bounded below (they are nonnegative), the sequence of values of G^R at these right endpoints converges, and we define $G^R(q_\ell)$ as its limit, and also set $G^R(q)=G^R(q_\ell)$ for q in in the interior of any maximal open interval (q_ℓ,q_r) on which b is constant. This concludes the construction of G^R for all quantities strictly lower than the maximum possible quantity; at this quantity we set G^R to equal 1. Thus G^R is a c.d.f. iff it is monotonic.

To establish monotonicity, suppose that q_ℓ, q_r are such that $q_\ell<q_r$, $G^R(q_\ell)\leq G^K(q_\ell)$, and

[93] An interval is maximal with respect to a given property if there is no larger, inclusive interval that also satisfies the property.

[94] Note that G^R is well defined if it so happens that $q_r=\tilde{q}_\ell$.

[95] Notice that the limit might be over right end-points of both strictly decreasing b and constant b intervals. We of course allow for a constant sequence, that is the case where q_ℓ is the right end point of an adjacent interval.

that b is strictly decreasing on (q_ℓ, q_r). Then on (q_ℓ, q_r), the pointwise first-order optimality conditions obtain, and we have

$$-\left(v\left(q\right)-b\left(q\right)\right)G_q^R\left(q\right)=\left(1-G^R\left(q\right)\right)b_q\left(q\right), \text{ and } -\left(v\left(q\right)-b\left(q\right)\right)G_b^K\left(q\right)=1-G^K\left(q\right);$$

in particular, G^R and G^K are continuous on (q_ℓ, q_r). The left-hand equation holds by construction of G^R and the right-hand equation follows from the fact that b is a best response to supply-reserve distribution K and opponent demand Q. By construction, the $-(v(q)-b(q))$ terms are equal, and so for any $q \in (q_\ell, q_r)$ it must be that

$$\frac{G_q^R\left(q\right)}{1-G^R\left(q\right)}=\frac{G_b^K\left(q\right)b_q\left(q\right)}{1-G^K\left(q\right)}. \tag{14}$$

Suppose that there is $q \in (q_\ell, q_r)$ such that $G^R(q) > G^K(q)$. Then there is $\hat{q} \in (q_\ell, q)$ such that $G^R(\hat{q}) = G^K(\hat{q})$, because the c.d.fs G^R and G^K are continuous on (q_ℓ, q_r) and $G^R(q_\ell) \leq G^K(q_\ell)$. At this $\hat{q}$, equation 14 becomes $G_q^R(\hat{q}) = G_b^K(\hat{q})b_q(\hat{q})$, and substituting in for equations 12 gives

$$G_q^R\left(\hat{q}\right)=G_b^K\left(\hat{q}\right)b_q\left(\hat{q}\right)=G_q^K\left(\hat{q}\right)-\mathbb{E}_\xi\left[K_q\left(q+Q\left(b\left(q\right);\xi\right)\right)\right]\leq G_q^K\left(\hat{q}\right).$$

We conclude that $G^K(\hat{q}) = G^R(\hat{q})$ implies $G_q^K(\hat{q}) > G_q^R(\hat{q})$, contradicting $G^R(q) > G^K(q)$. From this it follows that if b is strictly decreasing on $[q_\ell, q_r]$ and $G^R(q_\ell) \leq G^K(q_r)$, then $G^R|_{q\in[q_\ell,q_r]} \succeq_{\text{FOSD}} G^K|_{q\in[q_\ell,q_r]}$, and, in particular, $G^R(q_r) \leq G^K(q_r)$. This inequality allows us to conclude that if q_r is the limit of left endpoints $\tilde{q}_\ell > q_r$ of maximal intervals, then $G^R(q_r)$ is weakly below the limit of $G\left(\tilde{q}_\ell\right)$ on this sequence. We can conclude that that G^R is monotonic and hence a cumulative distribution function such that $G^R \succeq_{\text{FOSD}} G^K$.

We now define the random reserve distribution F^R as follows: for any q, let $F^R(b(q)) = 1 - G^R(q)$. We construct b^R that is first-order optimal bid function with respect to F^R and such that $b^R \geq b$. Our construction is iterative: we begin with $b^{R0} = b$, then show how to compute $b^{R[t+1]}$ from b^{Rt} for any $t \geq 0$. Let Ω_t be the set of maximal constant intervals of b^{Rt}. For an interval $(q_\ell, q_r) \in \Omega_t$, let $\tilde{q}_r$ solve the ironed first-order optimality condition at bid level $b^{Rt}(q_r)$,[96]

$$\left(F^R\left(b^{Rt}\left(q_r\right)\right)-\lim_{q\searrow q_r}F^R\left(b^{Rt}\left(q\right)\right)\right)\left(v\left(\tilde{q}_r\right)-b^{Rt}\left(q_r\right)\right)=\left(b^{Rt}\left(q_r\right)-b^{Rt}\left(\tilde{q}_r\right)\right)F^R\left(b^{Rt}\left(\tilde{q}_r\right)\right).$$

Since $p = b^{Rt}(q_r)$ is a level at which b is constant, there is a mass point in F^R at $b^{Rt}(q_r)$,

[96] Measure-zero changes in bid do not affect utility. Therefore we assume, without loss of generality, that b^{Rt} is left continuous.

and the first-order ironing equation cannot be solved at $\tilde{q}_r < q_r$. It follows that $\tilde{q}_r \geq q_r$, and moreover that $b^{Rt}(\tilde{q}_r) \leq v(\tilde{q}_r)$. Then let $\tilde{\Omega}_t$ be the set of intervals $(q_\ell, \tilde{q}_r)$, where $(q_\ell, q_r) \in \Omega_t$ and $\tilde{q}_r$ is derived from q_r as above. We now define $b^{R[t+1]}$,

$$b^{R[t+1]}(q) = \begin{cases} \sup\left\{b^{Rt}(q_r) : q \in (q_\ell, \tilde{q}_r) \in \tilde{\Omega}_t\right\} & \text{if } \exists\, (q_\ell, \tilde{q}_r) \in \tilde{\Omega}_t \text{ with } q \in (q_\ell, \tilde{q}_r)\,, \\ b^{Rt}(q) & \text{otherwise.} \end{cases}$$

By construction, $b^{Rt} \leq b^{R[t+1]} \leq v$, and thus $b^{Rt} \to b^R$ for some b^R.[97] Where the limit b^R is strictly decreasing, it is equal to b and therefore satisfies the first-order conditions for optimality. When the limit b^R is constant, it satisfies the ironed first-order conditions for optimality by construction. It follows that b^R is first-order optimal. Finally, since $b = b^{R0}$ and $b^{Rt} \leq b^{R[t+1]}$ for all t, it must be that $b \leq b^R$.

Being weakly higher than b, the bid function b^R induces a realized quantity distribution $\tilde{G}^R$ that is weakly stronger than G^R (the distribution of realized quantity with reserve distribution F^R and bid b), which is in turn weakly stronger than G^K, and it follows that $\pi(b^R; \tilde{G}^R) \geq \pi(b; G^K)$. Since F^R implements b^R as a first-order optimal bid function, the lemma follows. □

Remark 5. When supply-reserve distribution K and aggregate supply Q are discontinuous, we adjust the first condition of the definition of a bidder's first-order optimality at points at which G^K is not differentiable and require at these points that the the left derivative in b (which always exists, since G^K is decreasing in b) satisfies[98]

$$-\left(v\left(q\right) - b\left(q\right)\right) G^i_{b-}\left(q; b\right) - \left(1 - G^K\left(q; b\right)\right) \geq 0.$$

This is the only adjustment in the definition; the old definition is unchanged at points of differentiability and where bids are flat. We follow the construction of G^R in the proof of Lemma 14 with two adjustments: (i) we substitute the left derivative G^i_{b-} for derivative G^i_b, and (ii) the differential part of the construction is separately conducted for maximal intervals (q_ℓ, q_r) on which b is strictly decreasing and continuous (as opposed to merely strictly decreasing). In this way, we are able to construct G^R for all relevant quantity and price pairs, subject to verifying monotonicity like in the above proof of Lemma 14.

[97]Note that in the simple case where the original bid function b is strictly decreasing, it is the case that $b^R = b$. The iterative process applied here handles the possible need to extend to the right the constant intervals from the original bid function b, as well as the possibility that one constant interval "overtakes" another in the iterative process. Note that in the latter case $b^R(q) > b(q)$ for q in the overtaken constant interval of b.

[98]The left derivative of a function h at x is defined as $h_{x-}(x) = \lim_{\varepsilon \searrow 0}(h(x) - h(x - \varepsilon))/\varepsilon$. Similarly the right derivative equals $h_{x+}(x) = \lim_{\varepsilon \searrow 0}(h(x + \varepsilon) - h(x))/\varepsilon$.

The monotonicity continues to hold because G^K is monotonic and whenever b is strictly decreasing and continuous, we have

$$0 = -\left(v\left(q\right) - b\left(q\right)\right)\frac{G_q^R\left(q\right)}{b_{q+}\left(q\right)} - \left(1 - G^R\left(q\right)\right) \leq -\left(v\left(q\right) - b\left(q\right)\right) G_{b-}^K\left(q; b, Q\right) - \left(1 - G^K\left(q; b, Q\right)\right). \tag{15}$$

For any maximal interval (q_ℓ, q_r) on which b is continuous and strictly decreasing we prove monotonicity by contradiction, as before. If there is $q \in (q_\ell, q_r)$ such that $G^R(q) > G^K(q)$, there is $\hat{q} \in [q_\ell, q_r]$ such that $G^R(\hat{q}) = G^K(\hat{q})$: even though G^K is potentially discontinuous, G^R is guaranteed to be continuous on the maximal interval in question (it is the solution to a differential equation) and G^K is monotone. At this $\hat{q}$, plugging equations 12 into inequality 15 gives

$$G_{b-}^K\left(\hat{q}\right) \leq \frac{G_q^R\left(\hat{q}\right)}{b_{q+}\left(\hat{q}\right)}.$$

Since b is decreasing in q, this gives

$$\begin{aligned} G_q^R\left(\hat{q}\right) &\leq G_{b-}^K\left(\hat{q}\right) b_{q+}\left(\hat{q}\right) \\ &= G_{q+}^K\left(\hat{q}\right) - \mathbb{E}_\xi\left[K_{q+}\left(q + Q\left(b\left(q\right); \xi\right)\right)\right] \leq G_{q+}^K\left(\hat{q}\right). \end{aligned}$$

The final inequality follows from the fact that the exogenous supply-reserve distribution K satisfies $K_{q+} \geq 0$. Then $dG^R(q; b, Q)/dq \leq dG^K(q; b, Q)/dq+$ at $q = \hat{q}$, contradicting $G^R(q) > G^K(q)$ for some $q > \hat{q}$. The remainder of the proof follows the same steps as the original proof of Lemma 14.

G.2.3 Approximation by Strictly-Decreasing Bid Functions

We now show that we can arbitrarily approximate the first-order optimal bid b^R associated with random reserve F^R with a *strictly* decreasing bid function $\tilde{b}^R$, associated with some random reserve distribution $\tilde{F}^R$, and that the distribution of realized quantity under this approximation approximates the distribution of quantity under b^R. Then since $b^R \geq b$ and $\tilde{b}^R \approx b^R$, it follows that $\tilde{b}^R$ approximates the revenue generated by b under reserve distribution F^R arbitrarily closely, or yields higher revenue.

Lemma 15. *Given a reserve distribution F^R with first-order optimal bid b^R and any $\lambda > 0$, there is a reserve distribution $\tilde{F}^R$ with a strictly decreasing first-order optimal bid $\tilde{b}^R$such that $\pi(\tilde{b}^R; \tilde{G}^R) > \pi(b^R, G^R) - \lambda$.*

Proof. If b^R is strictly decreasing the claim is trivially satisfied. Therefore, assume that b^R is constant on the (maximal) interval (q_ℓ, q_r). Let $\tilde{b}^R \leq b^R$ be strictly decreasing on (q_ℓ, q_r) and such that $\tilde{b}^R|_{q\notin(q_\ell,q_r]} = b^R|_{q\notin(q_\ell,q_r]}$ and $\tilde{b}^R(q_r) = \lim_{q'\searrow q_r} b^R(q')$. Let $\tilde{F}^R|_{p\geq b^R(q_\ell)} = F^R|_{p\geq b^R(q_\ell)}$. Then $\tilde{b}^R$ is a first order best response for all $p \geq b^R(q_\ell)$ because the definition of first order optimality is point-wise.

We now show that $\tilde{b}^R$ can be specified on $(q_\ell, q_r]$ so that (i) the probability that $q \in (q_\ell, q_r]$ is lower under $\tilde{b}^R$ than under b^R (thus the probability that $q > q_r$ is higher under $\tilde{b}^R$ than under b^R), (ii) $\tilde{b}^R$ is relatively close to b^R, and (iii) the conditional revenue under $\tilde{b}^R$, given $q \in (q_\ell, q_r]$, is not significantly below the conditional revenue under b^R. First, for a distribution F let $\Delta F \equiv F(\tilde{b}^R(q_\ell)) - F(\tilde{b}^R(q_r))$; since $\tilde{b}^R$ is first-order optimal and is strictly decreasing on $[q_\ell, q_r]$,

$$\begin{aligned}
\Delta\tilde{F}^R &= \left[\exp\left(\int_{\tilde{b}^R(q_r)}^{\tilde{b}^R(q_\ell)} \frac{1}{v\left(\tilde{\varphi}^R\left(y\right)\right) - y} dy\right) - 1\right] \tilde{F}^R\left(\tilde{b}^R\left(q_r\right)\right) \\
&< \left[\exp\left(\ln\left[v\left(q_r\right) - \tilde{b}^R\left(q_r\right)\right] - \ln\left[v\left(q_r\right) - \tilde{b}^R\left(q_\ell\right)\right]\right) - 1\right] \tilde{F}^R\left(\tilde{b}^R\left(q_r\right)\right) \\
&= \left(\frac{\tilde{b}^R\left(q_\ell\right) - \tilde{b}^R\left(q_r\right)}{v\left(q_r\right) - \tilde{b}^R\left(q_\ell\right)}\right) \tilde{F}^R\left(\tilde{b}^R\left(q_r\right)\right) = \left(\frac{\tilde{F}^R\left(\tilde{b}^R\left(q_r\right)\right)}{F^R\left(\tilde{b}^R\left(q_r\right)\right)}\right) \Delta F^R. \qquad (16)
\end{aligned}$$

The first inequality follows from the fact that v and $\tilde{\varphi}^R$ are strictly decreasing, and the final equality follows from the fact that b^R is first-order optimal with respect to F^R and is flat on $[q_\ell, q_r]$. Now suppose that $\tilde{F}^R(\tilde{b}^R(q_r)) < F^R(\tilde{b}^R(q_r))$; by inequality (16) it must be that $\Delta\tilde{F}^R < \Delta F^R$, and since $\tilde{F}^R(\tilde{b}^R(q_\ell)) = F^R(\tilde{b}^R(q_\ell))$ it follows that $\tilde{F}^R(\tilde{b}^R(q_r)) > F^R(\tilde{b}^R(q_r))$, a contradiction. Then $\tilde{F}^R(\tilde{b}^R(q_r)) \geq F^R(\tilde{b}^R(q_r))$, implying directly that $\Delta\tilde{F}^R \leq \Delta F^R$. Thus point (i) holds for any $\tilde{b}^R$.

Points (ii) and (iii) are shown by construction. For $\delta > 0$ sufficiently small, let $\tilde{b}^R(q_r - \delta) > \tilde{b}^R(q_\ell) - \delta$. Since $\tilde{F}^R|_{p>\tilde{b}^R(q_\ell)} = F^R|_{p>\tilde{b}^R(q_\ell)}$, the expected revenue generated by bid $\tilde{b}^R$ under distribution $\tilde{F}^R$, conditional on $p > \tilde{b}^R(q_\ell)$, is identical to the expected revenue generates by bid b^R under distribution F^R, conditional on $p > \tilde{b}^R(q_\ell)$. Letting $\tilde{b}^R|_{p<\tilde{b}^R(q_r)} = b^R|_{p<\tilde{b}^R(q_r)}$, we have $||\tilde{b}^R - b^R|| < (q_r - q_\ell)\delta + (\tilde{b}^R(q_\ell) - \tilde{b}^R(q_r))\delta$ by construction. By point (i) and the analysis in the proof of Lemma 14, $\tilde{F}^R|_{p<\tilde{b}^R(q_r)} \preceq_{\text{FOSD}} F^R|_{p<\tilde{b}^R(q_r)}$, and so the expected revenue generated by bid $\tilde{b}^R$ under distribution $\tilde{F}^R$, conditional on $p < \tilde{b}^R(q_r)$, is $O(\delta)$ lower than the expected revenue generated by bid b^R under distribution F^R, conditional on $p < \tilde{b}^R(q_r)$. Finally, the utility lost when $p \in [\tilde{b}^R(q_r), \tilde{b}^R(q_\ell)]$ may be bounded in the following way. When $p \in [\tilde{b}^R(q_r), \tilde{b}^R(q_r) - \delta]$ at most quantity δ is lost (versus bid b^R), with marginal utility at most $\overline{v}$; this loss is incurred with at most probability 1, so this loss is bounded above by $\overline{v}\delta$. When $p \in [\tilde{b}^R(q_\ell) - \delta, \tilde{b}^R(q_\ell)]$, the quantity lost (versus bid b^R) is at most $(q_r - q_\ell) < \overline{Q}$, with marginal utility at most $\overline{v}$. However, the probability that this quantity is lost is bounded by

$$\begin{aligned}
&\tilde{F}^R\left(\tilde{b}^R\left(q_\ell\right)\right)-\tilde{F}^R\left(\tilde{b}^R\left(q_\ell\right)-\delta\right)\\
&=\left[\exp\left(\int_{\tilde{b}^R(q_\ell)-\delta}^{\tilde{b}^R(q_\ell)}\frac{1}{v\left(\tilde{\varphi}^R\left(y\right)\right)-y}dy\right)-1\right]\tilde{F}^R\left(\tilde{b}^R\left(q_\ell\right)-\delta\right)\\
&\leq\left[\exp\left(\int_{\tilde{b}^R(q_\ell)-\delta}^{\tilde{b}^R(q_\ell)}\frac{1}{v\left(q_r\right)-y}dy\right)-1\right]\tilde{F}^R\left(\tilde{b}^R\left(q_\ell\right)\right)\\
&=\left[\exp\left(\ln\left[v\left(q_r\right)-\left(\tilde{b}^R\left(q_\ell\right)-\delta\right)\right]-\ln\left[v\left(q_r\right)-\tilde{b}^R\left(q_\ell\right)\right]\right)-1\right]\tilde{F}^R\left(\tilde{b}^R\left(q_\ell\right)\right)\\
&=\left(\frac{\delta}{v\left(q_r\right)-\tilde{b}^R\left(q_\ell\right)}\right)\tilde{F}^R\left(\tilde{b}^R\left(q_\ell\right)\right).
\end{aligned}$$

Then this probability is also bounded above by a term linear in δ.[99] Then for any $\lambda > 0$ there is $\delta > 0$ such that the revenue generated by the first-order optimal bid function $\tilde{b}^R$ under reserve distribution $\tilde{F}^R$ is no more than λ below the revenue generated by the first-order optimal bid function b^R under reserve distribution F^R. □

The above two lemmas imply the following approximation result:

Lemma 16. *Given any best response bid curve $b(\cdot)$ and any $\lambda > 0$, there is a massless reserve distribution $\tilde{F}^R$ with strictly decreasing first-order best response $\tilde{b}^R$ such that such that the first order best response to F^R generates no more than λ less revenue than $b(\cdot)$.*

G.2.4 An Auxiliary Uniform-Price Auction with Known Values

We maintain the auxiliary assumption that the bidder whose response we analyze has no private information. Having shown that we can restrict attention to random reserve, we continue the analysis by showing that any strictly decreasing first-order optimal bid $\tilde{b}^R$generates strictly less revenue than some uniform-price auction (Theorem 17), which we then bound by pay-as-bid revenue in the next and final subsection, where we also drop the no-private-information assumption.

Lemma 17. [Uniform-Price Revenue Implementation] *Given a massless distribution of reserve prices F^R and a strictly decreasing first-order optimal bid b^R, there is a distribution of reserve prices $\hat{F}^R$ such that the uniform-price auction under reserve distribution $\hat{F}^R$ generates the same expected revenue as the pay-as-bid auction with first-order optimal bid b^R and reserve distribution F^R.*

[99]Since $b(\cdot) < v(\cdot)$ for all units which are received with strictly positive probability (Lemma 9), $v(q_r) - b^R(q_r) = v(q_r) - \tilde{b}^R(q_\ell) > 0$.

While the above lemma shows that uniform-price can match the revenue of pay-as-bid, we need to bear in mind that it is an auxiliary result in which we assumed that the bidder analyzed has no private information.

Proof. We may assume that the support of the distribution F^R is contained in the support of marginal values on units the bidder can win. Indeed, our assumptions on the utility imply that this support is convex and thus reserves outside of support are either above or below it. The mass of reserve prices above the support can be arbitrarily shifted to reserves in the support, increasing expected revenue. The mass of reserves below the support can be shifted to the minimum of the support, again weakly increasing the revenue. The latter operation might create an atom at the bottom of the support, but as we have seen in the proofs for Section 3 (cf. Appendix C.5), this atom does not affect the bidder's best response behavior. Under these assumptions, truthful reporting, $b \equiv v$, is the essentially unique equilibrium in a uniform-price auction with random reserve drawn from F^R. Under a random reserve distribution, each bidder's problem is a single-person decision problem. Because demand at a particular price does not affect outcomes at other prices, at each price bidders should demand a utility-maximizing quantity. Thus at each p, $v(\hat{\varphi}^R(p)) = p$.[100]

Revenue in the pay-as-bid auction under reserve distribution F^R is

$$\mathbb{E}[\pi] = \int_{\underline{b}}^{\overline{b}} \left(p\varphi^R(p) + \int_p^{\overline{b}} \varphi^R(x)\,dx \right) f^R(p)\,dp.$$

Define $\hat{F}^R$ so that

$$\hat{F}^R\left(v\left(\varphi^R(p)\right)\right) = F^R(p;s).$$

By construction, $\hat{F}^R_p(v(\varphi^R(p)))v_q(\varphi^R(p))\varphi^R_p(p) = F^R_p(p)$. Additionally, $\operatorname{Supp}\hat{F}^R = [\underline{p},\overline{v}]$, and in a uniform-price auction with reserve distribution $\hat{F}^R$, it is weakly optimal for the bidder to submit truthful bids for all quantities q such that $v(q) \in [\underline{b},\overline{v}]$. The revenue in this auction is

$$\mathbb{E}[\hat{\pi}] = \int_{\underline{b}}^{\overline{v}} pv^{-1}(p)\,\hat{F}^R_p(p)\,dp.$$

Apply a change of variables, so that $p = \hat{v}(\varphi^R(p'))$. Then $dp = v_q(\varphi^R(p'))\varphi^R_p(p')dp'$. Since

[100] Since b is strictly decreasing and first-order optimal, φ and φ_p are well-defined for all feasible prices p.

$\varphi^R(\overline{p}) = 0$, this gives

$$\begin{aligned}
\mathbb{E}\left[\hat{\pi}\right] &= \int_{\underline{b}}^{\bar{b}} v\left(\varphi^R\left(p'\right)\right) v^{-1}\left(v\left(\varphi^R\left(p'\right)\right)\right) \hat{F}^R_p\left(v\left(\varphi^R\left(p'\right)\right)\right) v_q\left(\varphi^R\left(p'\right)\right) \varphi^R_p\left(p'\right) dp' \\
&= \int_{\underline{b}}^{\bar{b}} v\left(\varphi^R\left(p'\right)\right) \varphi^R\left(p'\right) F^R_p\left(p'\right) dp'.
\end{aligned}$$

Then compare,

$$\begin{aligned}
\mathbb{E}\left[\pi\right] - \mathbb{E}\left[\hat{\pi}\right] &= \int_{\underline{b}}^{\bar{b}} \left(p\varphi^R\left(p\right) + \int_p^{\bar{b}} \varphi^R\left(x\right) dx\right) F^R_p\left(p\right) - v\left(\varphi^R\left(p\right)\right) \varphi^R\left(p\right) F^R\left(p\right) dp \\
&= \int_{\underline{b}}^{\bar{b}} \left(-\left(v\left(\varphi^R\left(p\right)\right) - p\right) \varphi^R\left(p\right) + \int_p^{\bar{b}} \varphi^R\left(x\right) dx\right) F^R_p\left(p\right) dp \\
&= \int_{\underline{b}}^{\bar{b}} \left(-\left[\frac{F^R\left(p\right)}{F^R_p\left(p\right)}\right] \varphi^R\left(p\right) + \int_p^{\bar{b}} \varphi^R\left(x\right) dx\right) F^R_p\left(p\right) dp \\
&= -\int_{\underline{b}}^{\bar{b}} \varphi^R\left(p\right) F^R\left(p\right) dp + \int_{\underline{b}}^{\bar{b}} \int_p^{\bar{b}} \varphi^R\left(x\right) dx F^R_p\left(p\right) dp \\
&= -\int_{\underline{b}}^{\bar{b}} \varphi^R\left(p\right) F^R\left(p\right) dp + \left[\int_p^{\bar{b}} \varphi^R\left(x\right) dx F^R\left(p\right)\right]\Bigg|_{p=\underline{b}}^{\bar{b}} + \int_{\underline{b}}^{\bar{b}} q^R\left(p\right) F^R\left(p\right) dp = 0.
\end{aligned}$$

The transition from the second line to the third comes from the bidder's first-order condition under random reserve. Then the uniform-price auction with reserve distribution $\hat{F}^R$ generates the same revenue as the pay-as-bid auction with reserve distribution F^R and first-order optimal bid b^R. □

G.2.5 Revenue Dominance of Deterministic Mechanisms with Known Values

Our previous lemmas imply that, when a bidder has no private information, the seller can weakly improve the revenue obtained from this bidder by implementing a uniform price auction with a random reserve price. These results are independent of opponent strategies in the pay-as-bid auction. Furthermore, we argued above that when the bidder participates in an auction with a random reserve price (and sufficiently large fixed supply) her best response is independent of her opponents' strategies. Thus, if the seller knew each bidder's private information, he could improve his revenue by implementing a bidder-specific uniform price auction with a random reserve price.

We are now ready to conclude the proof of Lemma 5 by showing that the above uniform price auction generates less revenue than a deterministic pay-as-bid auction, still in the auxiliary environment in which bidders have no private information, or as we may also express it, when their information is known to the seller.

Proof. Focusing on one bidder and putting together Lemmas 14, 15, and 17 we can conclude that for any $\lambda > 0$ and any random elastic supply in a pay-as-bid auction, there is a uniform-price auction with random reserve that raises from the bidder we focus on at least the pay-as-bid auction revenue minus λ. As we have seen in the first paragraph of the proof of Lemma 17, in this uniform-price auction we may assume that the bidder bids his or her marginal values (at all prices in the support of the random reserve distribution), and ex post revenue is always weakly below monopoly revenue. It follows that the uniform-price auction's revenue is maximized by selling the deterministic monopoly quantity with an appropriate reserve price. By Theorem 5, this revenue is equivalent to what the seller would obtain by implementing a pay-as-bid auction for the (deterministic) monopoly quantity, with or without a reserve price. Thus, to maximize the revenue obtained from a single bidder whose information is known to the seller, it is optimal to deterministically sell the bidder the monopoly quantity.

Because bidders are symmetric, it follows that it is optimal to deterministically sell them the aggregate monopoly quantity (note that the equilibrium price will be the monopoly price as long as the seller sets the reserves weakly below it). □

G.3 Proof of Theorem 15 (Optimality of Deterministic Mechanisms)

Proof. If the seller knows the bidders' common signal s, the optimal quantity in a pay-as-bid auction is $Q^\star(s) \in \arg\max_{Q \leq Q^{\max}} Q\hat{v}(Q; s)$, and in the unique equilibrium of this pay-as-bid auction, $p^\star(Q^\star(s); s) = \hat{v}(Q^\star(s); s)$.[101] Let $\mathcal{Q} : \mathbb{R}_+ \to \mathbb{R}_+$ be a supply curve, where $\mathcal{Q}(p) = \inf\{Q^\star(s) \colon p^\star(s) > p\}$. Bidder values are regular, so $\mathcal{Q}$ is increasing. Then equilibrium in the pay-as-bid auction with supply curve $\mathcal{Q}$ is such that for any bidder signal s, $p(Q^\star(s); s) = \hat{v}(Q^\star(s); s)$, and revenue is maximized for each type independently. □

G.4 Proof of Theorem 16 (Revenue Dominance of Pay-as-Bid)

Proof. Consider the (deterministic) optimal supply curve derived in Theorem 15. Given this supply curve, there is an equilibrium of the uniform-price auction in which bidders submit truthful bids. As in the pay-as-bid auction, for any realization of the bidder-common signal s the market clearing price and quantity corresponds to the monopoly solution, and revenue in this equilibrium of the uniform-price auction is equivalent to revenue in the unique

[101]Equilibrium uniqueness is established in Theorem 14.

equilibrium of the optimal pay-as-bid auction. No higher revenue is feasible in the uniform-price auction—even with different distribution over supply-reserve—because for known s the revenue is bounded above by monopoly revenue. $\square$