Prices and Efficient Assignments Without Transfers

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Abstract

We study the assignment of objects in environments without transfers allowing for both single-unit and general multi-unit demands, covering a wide range of applied environments, from school choice to course allocation. We establish the Second Welfare Theorem for these environments despite them failing the local non-satiation condition that previous studies of the Second Welfare Theorem relied on. We also prove a strong-version of the First Welfare Theorem. We thus show that the link between efficiency and decentralization through prices is valid in environments without transfers.

1 Introduction

A classic insight of the Walrasian theory of markets, commonly referred to as the Second Welfare Theorem, is that every Pareto efficient assignment can be decentralized through the use of prices.\(^1\) This classic insight is predicated on the assumption that agents are locally non-satiated, an assumption that is readily satisfied in settings with money.\(^2\) However, the

\(^1\)This classic insight, also known as the Second Fundamental Theorem of Welfare Economics, was conjectured by Pareto (1909), and subsequently refined and developed by many authors, culminating in the definitive treatment by Arrow (1951) and Debreu (1951).

\(^2\)Local non-satiation requires that for any agent and any assignment there is a nearby assignment that the agent strictly prefers, for instance because it leaves him with more money.
non-satiation assumption fails in settings without money such as the assignment of school seats in school choice programs. The assignment of scarce resources in these settings has been intensively studied recently and Pareto efficiency is a commonly accepted goal for such assignments.\(^3\)

What assignments are efficient in settings without transfers? Does the insight of the Second Welfare Theorem remain valid in such settings? We address these questions in the canonical no-transfer assignment model of Hylland and Zeckhauser (1979). There is a finite set of agents and objects. Each agent’s utility is given by their von Neumann-Morgenstern valuations, and agents evaluate lotteries according to the expected utility theory.\(^4\) Following Hylland and Zeckhauser, we study Walrasian equilibria in which each agent is endowed with token money; the amount of token money held after the assignment has no impact on agents’ utilities.

We establish the Second Welfare Theorem in this environment: despite the lack of transfers and the possibility of satiation, every efficient assignment may be supported in a Walrasian equilibrium that is decentralized via prices, just as Pareto efficient assignments in environments with transfers can be. In market design contexts, our characterization of efficient assignments allows one to restrict attention to price mechanisms at least in settings, such as large markets, where such mechanisms are incentive compatible.\(^5\) In particular, every Pareto efficient assignment may be implemented as an outcome of Hylland and Zeckhauser’s mechanism with properly chosen budgets.

The problems the received approach to the Second Welfare Theorem runs into in settings with locally satiated agents are well-known (Mas-Collel, Winston, and Green, 1995), and hence it is rather surprising that the insight of the Second Welfare Theorem holds true in the canonical no-transfer environment we study.\(^6\) Indeed, whether the Second Welfare Theorem obtains in settings without transfers and with possibly satiated agents remained a puzzle except for deterministic assignments. For deterministic assignments, Abdulkadiroglu and Sonmez (1998) established a version of the Second Welfare Theorem; they showed that each

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\(^3\)See e.g. Abdulkadiroglu and Sonmez (2003).

\(^4\)We first analyze the case where each agent demands at most one object, as in school seat assignment, and we then extend our results to the setting in which agents may demand multiple objects.

\(^5\)The large market incentive compatibility of the mechanisms that set the prices endogenously has been established by He et all (2018) and Azevedo and Budish (2019). Price mechanisms are also incentive compatible in settings in which we can set prices exogenously, for instance when we have at least an approximate sense of the distribution of preferences, see Pycia (2014).

\(^6\)The First Welfare Theorem also obtains in our environment. It was established by Hylland and Zeckhauser (1979), and further refined by Mas-Collel (1992) and Budish, Che, Kojima, and Milgrom (2013). For instance, all equilibria are efficient if agents strictly rank any two objects. There are, of course, many environments in which the First Welfare Theorem holds true, and the Second Welfare Theorem fails, see Mas-Collel et al (1995).
deterministic assignment may be obtained via a serial dictatorship. Every serial dictatorship can be implemented via budgets and prices, but there are many efficient allocations that cannot be implemented via serial dictatorships; as we show they all can be implemented via prices.\(^7\)

In order to prove the Second Welfare Theorem we develop a new approach to the proof because the failure of local non-satiation implies that the Separating Hyperplane Theorem commonly used to prove the Second Welfare Theorem guarantees only the existence of a separating hyperplane that may have non-empty intersections with the set of Pareto-dominant aggregate assignments.\(^8\) Facing the resulting prices, some agents might afford to buy bundles they strictly prefer over their assignment; this situation is called a quasi-equilibrium. In contrast, we prove the existence of a separating hyperplane that is disjoint with the set of Pareto-dominant aggregate assignments. Facing the resulting prices, no agent can afford a bundle they would prefer over their assignment, and the prices support the assignment as an equilibrium. Our approach is based on the theory of polytopes, and to the best of our knowledge ours is the first paper to leverage the properties of the polytopes to analyze Walrasian equilibria and prove the Second Welfare Theorem.\(^9\) As part of our proof, we establish a Full Separation Lemma for Polytopes that might be useful beyond the confines of our Walrasian analysis.

We show that our insight remains valid beyond the canonical school choice environment with single-unit demands, in a multi-unit random assignment model in which agents receive lotteries over bundles of indivisible goods; the number of objects in each bundles is bounded by agents’ capacity quota. In this environment, allocations that are Pareto efficient in the standard sense of being undominated by any feasible random assignment may be unimplementable through competitive equilibria.\(^{10}\) However, the Second Welfare Theorem obtains for allocations that are strongly efficient in the following sense: they are undominated by random allocations that are any feasible in expectation. Importantly, we prove that the

\(^7\) Random allocations cannot be implemented by deterministic serial dictatorships. As pointed out by Bogomolnaia and Moulin (2001), randomization over serial dictatorship is not necessarily efficient. Furthermore, notice that serial dictatorships elicit only agents' ordinal information, and as shown by Abdulkadiroglu, Che, and Yasuda (2011) mechanisms eliciting only ordinal information may lead to inefficient outcomes; this inefficiency does not necessarily vanish as the market becomes large, as shown by Pycia (2014).

\(^8\) While the full separation obtains if one of the separated sets is open, this assumption fails in our setting. Section 3 provides an example illustrating the failure of openness, and a more detailed discussion of why the standard techniques do not work.

\(^9\) For earlier uses of polytope ideas to study other questions in economics, see e.g. McLennan (2002), Budish et al (2013), Pycia and Unver (2015); none of these papers analyzes Walrasian equilibria.

\(^{10}\) The subtlety is caused by the failure of the Birkhoff-von Neumann theorem: in this environment random allocations whose expectations are feasible expectations may fail to be implementable as a lottery over feasible deterministic assignments. Cf. Nguyen, Peivandi and Vohra (2016) for a discussion of such failures of the Birkhoff-von Neumann theorem.
strong efficiency is not only sufficient but also necessary for the Second Welfare Theorem that is we also prove the analogue of the First Welfare Theorem for strong efficiency: every competitive equilibrium is efficient in the strong sense.\textsuperscript{11}

In the Online Appendix, we provide an independent analysis of the multi-unit-demand environment with \textit{perfectly divisible goods, possibly nonlinear preferences} and each agent demanding goods up to a capacity quota. In this general environment, we identify a sufficient condition for the Second Welfare Theorem to obtain.\textsuperscript{12}

Prior work on no-transfer assignments related price mechanisms to efficiency but only in conjunction with other strong requirements. In continuum economies, Thomson and Zhou (1993) related efficient, symmetric, and consistent mechanisms to Hylland and Zeckhauser’s mechanism, and Ashlagi and Shi (2014) showed that any efficient, symmetric, and strategy-proof random assignment can be expressed as the result of the Hylland and Zeckhauser mechanism.\textsuperscript{13} In contrast, we do not rely on symmetry, consistency, or strategy-proofness, and we prove our results for all finite economies.\textsuperscript{14}

Finally, we contribute to the literature on the Second Welfare Theorem beyond the standard exchange economy model. Anderson (1988) proved the Second Welfare Theorem for exchange economies with nonconvex preferences; in contrast with us, he maintained the assumption of local non-satiation. Florig and Rivera (2010) considered consumption bundles as indivisible at the individual level, yet perfectly divisible at the economy level. They define rationing equilibrium, of which a Walrasian equilibrium with fiat money is a particular case. They prove Welfare Theorems under the notion of weak Pareto-efficiency.\textsuperscript{15} Richter and Ru-

\textsuperscript{11}Our Second Welfare Theorem implies as a corollary that whenever feasibility in expectations is the relevant feasibility concept, then the Second Welfare Theorem holds true for standard Pareto efficiency. This is of relevance in large markets as Nguyen, Peivandi and Vohra (2016) extended the Birkhoff-von Neumann Theorem to multi-unit assignment in large markets showing that the set of feasible-in-expectations random assignments is asymptotically equivalent to the set of implementable random assignments.

\textsuperscript{12}We show that the Second Welfare Theorem is valid in this environment but only if agents’ preferences satisfy a "no marginal indifference" condition. This condition requires (for differentiable utility case) that the ranking of objects with respect to marginal utilities is strict and constant over the agent’s consumption space. Without this condition, the Second Welfare Theorem may fail. The online appendix analysis differs with our main model, in which e.g. we allow for indifferences. Also methodologically the appendix analysis is very different and builds on our construction of an algorithm that finds a "good" separating hyperplane in a finite number of steps (at most the number of object types minus 2), which to our knowledge is novel.

\textsuperscript{13}Makowski, Ostroy, and Segal (1999) showed a similar result for the classical exchange economies.

\textsuperscript{14}Related to our work are also other papers relying on the idea of using token money to allocate objects in the absence of transfers. Token money mechanisms have been extended beyond the canonical Hylland and Zeckhauser setting by, for instance, Sonmez and Uuver (2010), Budish (2011) and Budish and Cantillon (2012), c, Manjunath (2014), and Miralles (2016). Hafalir and Miralles (2015) analyze the utilitarian efficiency of such market approaches. None of these papers establishes a Second Welfare Theorem. Following on our work, Echenique, Miralles, and Zhang (2019) and Gul, Pesendorfer, and Zhang (2019) explored the First Welfare Theorem in the presence of constraints.

\textsuperscript{15}We thank a referee for pointing out this line of literature.
binstein (2014) propose a general convex geometry approach to welfare economics based on the concept of “primitive equilibrium,” where a strict linear ordering arranges alternatives in order to create “budget” sets. They prove a Second Welfare Theorem for the primitive equilibrium concept; when preferences are strictly monotone, their primitive equilibrium concept corresponds to the standard equilibrium concept, however, when specialized to our setting, this equilibrium concept becomes equivalent to the quasi-equilibrium discussed above.16

2 Base Model

We study a finite economy with agents \( i, j \in I = \{1, ..., |I|\} \) and indivisible objects \( x, y \in X = \{1, ..., |X|\} \). Each object \( x \) is represented by a number of identical copies \( j_x \in \mathbb{N} \). By \( S = (|x|)_{x \in X} \) we denote the total supply of object copies in the economy. If agents have outside options, we treat them as objects in \( X \); in particular, this implies that \( \sum_{x \in X} |x| \geq |I| \).

We assume initially that agents demand at most one copy of an object; we relax this assumption in Section 4. We allow random assignments and denote by \( q_i^x \in [0, 1] \) the probability that agent \( i \) obtains a copy of object \( x \). Agent \( i \)'s random assignment \( q_i = (q_i^1, ..., q_i^{|X|}) \) is a probability distribution. The economy-wide assignment \( Q = (q_i^x)_{i \in I, x \in X} \) is feasible if the aggregate assignment (which we will denote as \( A(Q) \)) is weakly lower that the supply vector: \( A(Q) \equiv \sum_{i \in I} q_i \leq S \). Let \( \mathcal{A} \) denote the set of economy-wide random assignments, and \( \mathcal{F} \subset \mathcal{A} \) denote the set of feasible random assignments. We call an assignment pure, or deterministic, if each of its elements \( q_i^x \) is either 0 or 1. By the Birkhoff-von Neumann theorem, a feasible random assignment can be expressed as a lottery over feasible pure assignments.

Agents are expected utility maximizers, and agent \( i \)'s utility from random assignment \( q_i \) equals the scalar product \( u_i(q_i) = v_i \cdot q_i \) where \( v_i = (v_i^x)_{x \in X} \in [0, \infty)^{|X|} \) is the vector of agent \( i \)'s von Neumann-Morgenstein valuations for objects \( x \in X \). We relax this linear utility form assumption also in Section 4.

We study the connection between two concepts: efficiency and equilibrium. A feasible random assignment \( Q^* \in \mathcal{F} \) is (ex-ante) Pareto efficient (or, simply, efficient) if no other feasible random assignment \( Q \in \mathcal{F} \) is weakly preferred by all agents and strictly preferred by some agents.

A random assignment \( Q^* \) and a price vector \( p^* \) constitute an equilibrium for a budget

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16 In Section 3 we provide an example of a quasi-equilibrium which is not an equilibrium; this quasi-equilibrium is a primitive equilibrium in the sense of Richter and Rubinstein. To the best of our knowledge the above discussion covers all extensions of the Second Welfare Theorem beyond the standard strictly monotone and convex setting. Of course, the literature on Walrasian equilibria beyond this setting is richer, and—in addition to the papers cited above (including in footnotes)—includes, for instance, Bergstrom (1976), Manelli (1991), and Hara (2005) who focused on equilibrium existence and core convergence rather than on the Second Welfare Theorem.
vector \( w^* \in \mathbb{R}_{+}^{I} \) if \( Q^* \) is feasible, \( p^* \cdot q_i^* \leq w_i^* \) for any \( i \in I \), and \( u_i(q_i) > u_i(q_i^*) \implies p^* \cdot q_i > w_i^* \) for any \((q_i)_{i \in I} \in \mathcal{A}\).

3 The Second Welfare Theorem for School Choice

We now develop the Second Welfare Theorem for the canonical school choice setting. In the next section, we use this school choice Second Welfare Theorem to derive a very general Second Welfare Theorem for assignment with multi-unit demand.

Theorem 1. (The Second Welfare Theorem in Random Unit Assignments) If \( Q^* \in \mathcal{F} \) is Pareto-efficient, then there is a vector of budgets \( w^* \in \mathbb{R}_{+}^{I} \) and a vector of prices \( p^* \in \mathbb{R}_{+}^{X} \) such that \( Q^* \) and \( p^* \) constitute an equilibrium with budgets \( w^* \).

Before laying out the proof, let us compare our problem to the standard second welfare theorem with transfers and preferences that are convex and strictly monotonic. The well-known argument in the standard setting relies on the celebrated separating hyperplane theorem and it goes as follows. The set of aggregate feasible assignments is convex, and, given an efficient assignment \( Q^* = (q_i^*)_{i \in I} \) we want to implement, the set of (infeasible) aggregate assignments that Pareto dominate \( Q^* \) is convex as well. Since these two sets are disjoint, the separating hyperplane theorem tells us there exists a hyperplane that separates them. The normal vector to this hyperplane gives us a price vector \( p^* \), and the separation means that

\[
u_i(q_i) > u_i(q_i^*) \implies p^* \cdot q_i \geq w_i^*
\]

for any \( i \in I \) and for any \((q_i)_{i \in I} \in \mathcal{A}\). In other words, the separating hyperplane theorem allows us to find a price vector \( p^* \) that implements the efficient assignment \( Q^* \) as a so-called quasi-equilibrium.

The final step of the standard proof is then to show that this quasi-equilibrium is in fact an equilibrium, that is

\[
u_i(q_i) > u_i(q_i^*) \implies p^* \cdot q_i > w_i^*
\]

for any \( i \in I \) and for any \((q_i)_{i \in I} \in \mathcal{A}\). This last step is by contradiction: we take an assignment \( Q = (q_i)_{i \in I} \) that Pareto dominates \( Q^* \) while costing the same as \( Q^* \); in the neighborhood of \( Q \) we then find an assignment that still Pareto dominates \( Q^* \) while being cheaper than it. This is a contradiction as in quasi-equilibrium no cheaper assignment can Pareto dominate \( Q^* \).

It is this final step of the standard proof that fails in our setting. The standard separating hyperplane theorem separates the Pareto dominating aggregate assignments from the feasible
Figure 1: The simplex of “full-consumption” aggregate assignments. Aggregate assignment $A(Q^*)$ is on the intersection of the boundaries of sets $Y$ and $Z$.

ones. Yet full separation is not guaranteed. Unlike in the standard setting, in the setting with locally satiated preferences and without transfers, not every quasi-equilibrium is an equilibrium. The reason is as follows. In the standard setting with strongly monotone preferences no good can have a price of zero since agents would demand an infinite amount of such a good. In contrast, zero prices are the staple of our setting as recognized already by Hylland and Zeckhauser (1979). In particular, in a quasi-equilibrium an agent may be assigned a zero-price object while he strictly prefers another zero-price object. As an illustration consider the following example.

**Example 1.** Consider an economy with four agents and three objects. Two of the agents have von Neumann-Morgenstern utility vector $v = (\frac{1}{2}, 0, 1)$, and the remaining two agents have the utility vector $v' = (0, 1, \frac{1}{2})$. Suppose that there are three copies of object 1, one copy of object 2, and one copy of object 3. The following allocation $Q^*$ is then Pareto-efficient: $v$-agents obtain $q^* = (\frac{1}{2}, 0, \frac{1}{2})$ and $v'$-agents obtain $q'^* = (\frac{1}{2}, 1, 0)$.

The resulting aggregate assignment $A(Q^*)$ is $(2, 1, 1)$. Figure 1 places this point in the barycentric simplex of aggregate assignments in which exactly four units are assigned, that is such that for each agent the sum of probabilities of the three goods is 1 (the full-consumption simplex). Set $Y$ represents feasible aggregate assignments in the simplex; it is the triangle spanned by $(2, 1, 1), (3, 0, 1)$ and $(3, 1, 0)$. Set $Z$ represents all aggregate assignments $A(Q)$ in the simplex such that there exists an assignment $Q$ in which all agents are weakly better-off than under $Q^*$ and at least one agent is strictly better-off, and such that $A(Q)$ is the aggregate assignment of $Q$ (these assignments are, of course, not feasible). Set $Z$ has five corners:

- $(2,1,1)$, the aggregate assignment corresponding to $Q^*$,
- $(1,2,1)$, the aggregate assignment when $v$-agents obtain $q^*$ and $v'$-agents obtain $(0,1,0)$,
• \((0, 2\frac{1}{2}, 1\frac{1}{2})\), the aggregate assignment when \(v\)-agents obtain \((0, \frac{1}{3}, 2)\) and \(v'\)-agents obtain \((0, 1, 0)\),

• \((0, 0, 4)\), the aggregate assignment when each agent obtains good 3

• \((1, 0, 3)\), the aggregate assignment when \(v\)-agents obtain \(q\) and \(v'\)-agents obtain \((0, 0, 1)\).

Only the middle three corners belong to \(Z\), and one of the borders of \(Z\), the dashed line, is disjoint with \(Z\). In particular, the set \(Z\) is neither open nor closed.

Restricting attention to the assignments in the simplex, there is a horizontal hyperplane separating \(Y\) and \(Z\). This hyperplane corresponds to prices \(p^3 > p^2 = p^1 = 0\). When \(v\)-agents have budget \(\frac{1}{2}p^3\) and \(v'\)-agents have budget zero, these prices support \(Q^*\) as a quasi-equilibrium but not as an equilibrium. Indeed, \(v'\)-agents would rather buy a sure copy of object 2 than the lottery \(q''\), and both these outcomes have the price of zero.

We develop a new proof approach to establish the second welfare theorem and to address the difficulties illustrated in Example 1. To understand our approach observe that in Example 1, there are non-horizontal hyperplanes that fully separate \(Y\) and \(Z\) (in the full-consumption simplex). We show that this is always the case. A key step in the proof is the following new Separating Hyperplane Lemma that establishes that under conditions that—as we will shortly see—are always satisfied in the no-transfer assignment problem, full separation is possible.\(^{17}\)

**Lemma 1. (Full Separation Lemma)** Suppose \(Y \subset \mathbb{R}^n\) is a closed and convex polytope, suppose \(Z \subset \mathbb{R}^n\) is convex and non-empty, and \(\bar{Z} \subset \mathbb{R}^n\) is a closed and convex polytope containing \(Z\). Suppose further that \(Z \cap Y = \emptyset\) and that for all \(y \in Y \cap \bar{Z}\), \(\delta \in \mathbb{R}^n\), and \(\varepsilon > 0\) if \(y + \delta \in Z\), then \(y - \varepsilon \delta \notin \bar{Z}\). Then, there exists a price vector \(p \in \mathbb{R}^n_+\) and a budget \(w \in \mathbb{R}\) such that for any \(z \in Z\) and \(y \in Y\) we have \(p \cdot z > w \geq p \cdot y\) and such that for any \(\bar{z} \in \bar{Z}\) and \(y \in Y\) we have \(p \cdot \bar{z} \geq w \geq p \cdot y\).

We provide the proof of the lemma in Appendix A.

We can easily visualize the statement of the lemma in the context of Example 1. Both the set \(Y\) of feasible aggregate assignments and the set \(Z\) of (infeasible) aggregate assignments that Pareto dominate \(Q^*\) are polytopes. Our separation lemma says that if every line through 

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\(^{17}\)Let us mention a very elegant Polytope Separation Lemma that McLennan (2002) developed in an ordinal context unrelated to the problems studied in our paper, and that was never previously used to analyze Walrasian equilibria. McLennan’s lemma cannot be substituted for our Full Separation Lemma in the simple proof of our Second Welfare Theorem presented below because his lemma establishes only partial separation between polytopes, while our proof relies on full separation established by our lemma. (In December 2014 draft of our paper we developed an alternative proof of our Second Welfare Theorem in which we used McLennan’s lemma; this alternative proof required more complex conceptual structure than the direct proof.)
Q* and a point in Z has points that belong to the closure of Z only on one side of Q*, then there exists a fully separating hyperplane. The line assumption is satisfied in our example.

The rest of the proof of the second welfare theorem revolves around showing that indeed the assumption of the lemma is satisfied: no line through Q* can intersect the closure of Z on both sides of Q* (see the highlighted claim in the proof below).

**Proof of the Second Welfare Theorem.** For any random assignment Q ∈ A, we define the aggregate assignment A(Q) associated with Q to be \( \sum_{i \in I} q_i \), and we write Q ≻ Q* when \( u_i(q_i) \geq u_i(q^*_i) \) for every \( i \in I \) with at least one strict inequality.

Let Z = \{ A(Q) : Q ≻ Q*, Q ∈ A \}, and notice that the above assumption implies that Z is non-empty. Furthermore, Z is convex. Let \( \tilde{Z} = \text{Cl}(Z) \) be the topological closure of Z, and notice that \( \tilde{Z} \) is a non-empty convex polytope. Let Y = \{ A(Q) : Q ∈ F \} be the set of aggregate feasible random assignments. This set is a closed and convex polytope, and the efficiency of Q* implies that Z \( \cap Y = \emptyset \).

To use the full separation lemma, we need the following

**Claim.** For any \( y \in Y \cap \tilde{Z}, \delta \in \mathbb{R}^{|X|} \) and \( \varepsilon > 0 \), if \( y + \delta \in Z \) then \( y - \varepsilon \delta \notin \tilde{Z} \).

**Proof of the claim:** If \( y + \delta \in Z \) then there is a \( Q \succ Q^* \) such that \( A(Q) = y + \delta \). By way of contradiction, assume \( y - \varepsilon \delta \in \tilde{Z} = \text{Cl}(Z) \). Thus, there is a \( \tilde{Q} = (\tilde{q}_i)_{i \in I} \) such that \( u_i(\tilde{q}_i) \geq u_i(q^*_i) \) for every \( i \in I \) and \( A(\tilde{Q}) = y - \varepsilon \delta \). Then, the random assignment \( \tilde{Q} = \frac{\varepsilon}{1+\varepsilon}Q + \frac{1}{1+\varepsilon}\tilde{Q} \) is feasible, and the choice of Q and \( \tilde{Q} \) and the linearity of utility \( u_i(\cdot) \) in probabilities imply that \( \tilde{Q} \succ Q^* \). But this contradicts the fact that \( Q^* \) is ex-ante Pareto-efficient, proving the claim.

This claim and the full separation lemma imply that there exists a price vector \( p \in \mathbb{R}^{|X|} \) and a budget \( w \in \mathbb{R} \) such that \( p \cdot z > w \geq p \cdot y \), for any \( z \in Z \) and \( y \in Y \). Since \( Q^* \) is feasible \( \sum_{i \in I} q^*_i \in Y \) and thus \( p \cdot \sum_{i \in I} q^*_i \leq w \). Furthermore, \( p \cdot \sum_{i \in I} q^*_i \geq w \) because \( Q^* \in \text{Cl}(Z) \). We conclude \( p \cdot \sum_{i \in I} q^*_i = w \). Now, if we take some \( q_i \) that some agent \( i \in I \) strictly prefers to \( q^*_i \), then \( q_i + \sum_{j \in I \setminus \{i\}} q^*_j \in Z \), and we have \( p \cdot (q_i + \sum_{j \in I \setminus \{i\}} q^*_j) > w = p \cdot (q^*_i + \sum_{j \in I \setminus \{i\}} q^*_j) \). Consequently we have \( p \cdot q_i > p \cdot q^*_i \), proving that \( p \) and \( Q^* \) constitute an equilibrium for budgets \( w^*_i = p \cdot q^*_i \). QED

4 Multi-Unit Demand: Second and First Welfare Theorems

We now analyze the validity of our Second Welfare Theorem result in assignment economies in which participants demand multiple units of goods. As in the base model, we have a set
of agents $I$ and a set of objects $X$. Each object $x \in X$ has a finite number of copies $|x|$ and $S = (|x|)_{x \in X}$ is the supply vector. We relax the restriction that each agent demands at most one unit of goods and allow each agent to demand at most $k \in \mathbb{N}$ units of various goods, in total. We further assume that each agent receives exactly $k$ units. Both of these assumptions are without loss of generality because we allow objects that are supplied in large quantities but are worthless for the agents, called null objects, and because $k$ can be larger than the total supply of non-null objects.\footnote{To appreciate the generality of our model note that it allows one to accommodate constraints such as e.g. reserving some seats in a school to a group of applicants, while allowing all applicants to compete for the remaining seats.}

Let $B_i \subset \{0, 1, \ldots, k\}^{|X|}$ be the finite set of admissible individual bundles for agent $i$, and let $b_{i1}, \ldots, b_{i|B_i|}$ denote the elements of $B_i$. The set $B_i$ can accommodate any restrictions such as, for instance, that the agent consumes at most quantity 1 of each object. An individual random assignment $q_i \in \Delta(B_i)$ of agent $i \in I$ is a probability distribution over $B_i$. The agent’s expected utility is the scalar product $q_i \cdot v_i$ where $v_i \in \mathbb{R}^{|B_i|}$ is the vector of valuations for each bundle in $B_i$. For the sake of linear algebra calculations, we represent the set of bundles $B_i$ by the matrix $\beta_i = (b_{ib})_{x \in X, b \in B_i}$ in which $b_{ib} = 1$ iff object $x$ is part of bundle $b$, and $b_{ib} = 0$ otherwise.

A deterministic assignment of bundles $D = (b_i)_{i \in I} \in \times_{i \in I} B_i$ is feasible if $\sum_{i \in I} b_i \leq S$, coordinatewise. We denote by $\mathcal{D}$ the (finite) set of all feasible deterministic assignments of bundles and by $b_i(D)$ the bundle that agent $i$ obtains under the $D \in \mathcal{D}$. Denoting $B = \cup_i B_i$, a random assignment of bundles $Q = (q_i^b)_{i \in I, b \in B} \in [0, 1]^{I \times B}$ is \textbf{feasible in expectations} if each $q_i$ has support on $B_i$ and the expected aggregate assignment does not exceed supply for any good, $\sum_{i \in I, b \in B} q_i^b b \leq S$. A random assignment $Q = (q_i^b)_{i \in I, b \in B}$ is \textbf{feasible (or implementable)} if there are nonnegative weights $(\lambda_D)_{D \in \mathcal{D}} \geq 0$ summing up to 1 and such that, for every $i \in I$ and $b \in B$, $\sum_{b_i(D) = b} \lambda_D = q_i^b$. By $\mathcal{F}$ we denote the set of all feasible random assignments. Of course, every feasible assignment is feasible in expectations.\footnote{The assumptions that a deterministic assignment is feasible if $\sum_{i \in I} b_i \leq S$ and random assignment is feasible in expectations if $\sum_{i \in I, b \in B} q_i^b b \leq S$ are made for expositional simplicity only. Our results remain valid—with no changes in the proofs—for any other linear collection of constraints. Cf. also footnote 20.}
Our single-unit demand Second Welfare Theorem immediately implies the multi-unit demand Second Welfare Theorem if we allowed separate prices for all bundles. Indeed, then we can think of agents as having a single-unit demand: each of them demands at most one bundle.

The analysis becomes more subtle if we require—as in the definition of the competitive equilibrium above—that the price of a bundle is the sum of prices of the component goods of the bundle. We can then still apply our Full Separation Lemma and replicate the single-unit demand analysis provided every random assignment that is feasible in expectation is feasible. This property—established in the single unit case in the Birkhoff-von Neumann Theorem—is crucial for one of the conditions of our Full Separation Lemma: if we moved from an initial (feasible) aggregate assignment in some direction to a (non-feasible) Pareto-dominating aggregate assignment, then when moving in the opposite direction the assignments are not weakly Pareto dominant as otherwise a proper linear combination of both assignments would be feasible by the Birkhoff-von Neumann property and it would Pareto dominate the initial assignment.

There are multi-unit demand settings in which the Birkhoff-von Neumann property is true. Suppose for instance that each agent’s set of feasible bundles consists of exactly these bundles in which the quantity of each object is weakly below some respective quantity cup and in which agents’ utilities are additive: each agent $i$’s utility from a feasible bundle of objects is given by the sum of agent’s von Neumann-Morgenstern valuations $\bar{v}_i = (\bar{v}_i^1, \ldots, \bar{v}_i^{|X_i|})$ for objects in the bundle, that is the utility from bundle $q_i = (q_i^1, \ldots, q_i^{|X_i|}) \in X_i$ is the scalar product $q_i \bar{v}_i$; the utility from other bundles is zero. This additive utilities setting has been studied for instance by Budish et al (2013).\footnote{Budish et al. (2013) discuss how any profile of random assignments $\{q_i\}_{i \in I}$ that satisfies the above constraints can be implemented as lotteries over deterministic assignments. They also proved the First Welfare Theorem for the case of equal budgets and showed how to use Milgrom’s (2009) integer assignment messages to reduce certain non-linear preferences to this linear setting. The single-unit demand setting is the special case of the multi-unit demand setting, in which $|i| = 1$ for each agent $i$. As implied by our discussion of Birkhoff-von Neumann’s property, our Second Welfare Theorem remains true for any type of consumption constraints $X_i$ that satisfy Birkhoff-von Neumann’s property, e.g. because they satisfy Budish et al’s hierarchy condition or the conditions in Pycia and Unver (2015).}

At the same time, the Birkhoff-von Neumann Theorem does not in general extend to multi-unit assignments, as pointed out by Nguyen, Peivandi and Vohra (2016). The following example illustrates their point that for some infeasible assignments $Q = \{q_i\}_{i \in I} \notin \mathcal{F}$ the aggregate feasibility condition $\sum_{i \in I} \beta_i q_i \leq S$ might be satisfied. We use this example in further developments of this subsection.

**Example 2.** Consider the problem of assigning four objects $S = (1, 1, 1, 1)$ to two agents so
that each of them receives two objects. The set of possible bundles is

\[ B = \{(1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,0,1), (0,1,1,0), (0,0,1,1)\}. \]

The random assignment \( Q = (q_1, q_2) \) where \( q_1 = (1/2, 0, 0, 0, 0, 1/2) \) and \( q_2 = (0, 0, 1/2, 1/2, 0, 0) \) is feasible in expectation because \( \sum_{i \in \{1,2\}, b \in B} b q_i^b = S \). However, \( Q \) is not feasible. If there is \( (\lambda_D)_{D \in D} \geq 0 \), \( \sum_{D \in D} \lambda_D = 1 \) meeting the condition in the definition, there must be \( D \in D \) such that \( b_1(D) = (1,1,0,0) \) and \( \lambda_D > 0 \). However, \( \lambda_D > 0 \) implies that \( b_2(D) \in \{(1,0,0,1), (0,1,1,0)\} \). In either case \( D \) generates excess demand for either object 1 or object 2, contradicting \( D \in D \).

Where the Birkhoff-von Neumann property fails our previous analysis requires refinement: not only our proof approach but also the formulation of our Second Welfare Theorem. This is demonstrated by the following

**Proposition 1.** Not every efficient feasible random assignment \( Q^* \) is an equilibrium assignment.

**Proof.** Consider again the two agents and four objects from Example 4, with the set of feasible bundles studied in this example. Assume that \( v_1 = (1, 1 - \varepsilon, 0, 0, 1 - \varepsilon, 1) \) and \( v_2 = (0, 1 - \varepsilon, 1, 1, 1 - \varepsilon, 0) \) where \( \varepsilon \in (0, \frac{1}{2}) \). Consider assignment \( (q_1^*, q_2^*) \) such that \( q_1^* = (0, 1/2, 0, 0, 1/2, 0) \) and \( q_2^* = (0, 1/2, 0, 0, 1/2, 0) \) where the probabilities of bundles in \( B \) are listed in the same order as the bundles in Example 4. This assignment is feasible because we can implement it as a \(\frac{1}{2} : \frac{1}{2} \) lottery between two feasible deterministic assignments: \( ((1,0,1,0), (0,1,0,1)) \) and \( ((0,1,0,1), (1,0,1,0)) \).

The assignment \( (q_1^*, q_2^*) \) is also efficient. By way of contradiction, suppose that some other assignment \( (q_1, q_2) \) Pareto dominates \( (q_1^*, q_2^*) \). As the expected utility from the assignment \( Q^* \) is \( 1 - \varepsilon \) for both agents, we have

\[
q_1^1 + q_1^6 + (1 - \varepsilon) (q_1^2 + q_1^5) \geq 1 - \varepsilon, \\
q_2^2 + q_2^4 + (1 - \varepsilon) (q_2^2 + q_2^5) \geq 1 - \varepsilon,
\]

where superscripts on probabilities \( q_1^1, \ldots, q_1^6 \) denote the position in which the bundles are listed in \( B \). Denoting \( \rho_1 \equiv q_1^1 + q_1^6 = q_1^2 + q_1^5, \rho_2 \equiv q_1^3 + q_1^4 = q_2^3 + q_2^4, \) and \( \rho_3 \equiv q_1^2 + q_1^5 = q_2^2 + q_2^5, \) and recognizing that \( 1 - \rho_3 = \rho_1 + \rho_2 \), we can rewrite the above inequalities as

\[
\rho_1 \geq (1 - \rho_3) (1 - \varepsilon) = (\rho_1 + \rho_2) (1 - \varepsilon), \\
\rho_2 \geq (1 - \rho_3) (1 - \varepsilon) = (\rho_1 + \rho_2) (1 - \varepsilon).
\]
Because $\varepsilon < 1/2$, this system of inequalities cannot be satisfied unless $\rho_1 = \rho_2 = 0$. Hence, $(q_1, q_2)$ must put all the weight on the second and fifth bundle, just like $(q_1^*, q_2^*)$, and we can conclude that no feasible random assignment Pareto-dominating $(q_1^*, q_2^*)$.

In spite of being feasible and efficient, $(q_1^*, q_2^*)$ cannot be an equilibrium assignment. Indeed, for any vector of prices $p \in \mathbb{R}_+^{|X|}$ the cost of each of the bundles $q_1^*, q_2^*$, $q_1 = (1/2, 0, 0, 0, 0, 1/2)$, and $q_2 = (0, 0, 1/2, 1/2, 0, 0)$ is $\frac{1}{2} \sum_x p_x$, while $q_i \cdot v_i > q_i^* \cdot v_i$ for both $i \in \{1, 2\}$.

4.1 Second Welfare Theorem

In order to recover the Second Welfare Theorem we will strengthen the Pareto efficiency requirement. We say that a feasible random assignment of bundles $Q^*$ is **strongly efficient** if it is not ex-ante Pareto-dominated by any feasible-in-expectations random assignment of bundles. Because every feasible assignment is feasible in expectations, strong efficiency is indeed more demanding than efficiency we studied so far. A positive feature of strong efficiency, and an advantage over the efficiency concept studied above is that verifying it does not require the market participants to verify whether swaps of probabilities can be implemented; it is the natural concept when thinking in terms of marginal probabilities. In all settings that satisfy the Birkhoff-von Neuman Theorem, strong efficiency and efficiency are of course equivalent.

The following result then holds\(^{21}\)

**Theorem 2.** (Second Welfare Theorem for General Multi-unit Demands) If a feasible random assignment of bundles $Q^*$ is strongly efficient, then it is an equilibrium random assignment supported by some vector of prices $p^* \in \mathbb{R}_+^{|X|}$ and some vector of budgets $w^* = (w^*_i)_{i \in I} \in \mathbb{R}_+^{|I|}$.

We prove this theorem as an immediate corollary from the following

**Theorem 3.** If a feasible-in-expectations random assignment of bundles $Q^*$ cannot be ex-ante Pareto-dominated by any other feasible-in-expectations random assignment of bundles, then $Q^*$ is an equilibrium random assignment supported by some prices $p^* \in \mathbb{R}_+^{|X|}$ and budgets $(w^*_i)_{i \in I} \in \mathbb{R}_+^{|I|}$.

The latter result is more general because it only requires random assignment of bundles $Q^*$ to be feasible-in-expectation.

\(^{21}\)Combining this result and the previous proposition, we can conclude that in the setting of Example 4 efficiency does not imply strong efficiency.
To get a sense of the proof, notice that each random assignment over bundles determines the expected assignment of agent $i$ over the underlying goods, $\mu_i = \beta_i q_i$. Because the prices are defined on the underlying goods, every lottery over bundles that leads to the same expected assignment over the underlying goods has the same price. We can also input utility to the expected assignment by recognizing that in the equilibrium an agent buys the lottery over bundles in $B_i$ that maximizes the agent’s utility among all lotteries of the same price. For every expected assignment $\mu_i$ in the convex hull of $B_i$—the convex hull denoted by $Co(B_i)$—we thus define agent $i$’s utility $V_i$ from $\mu_i$ as

$$V_i(\mu_i) = \max_{\{q \in \Delta(B_i) | \beta_i q = \mu_i\}} q \cdot v_i.$$  

The following property of this utility function allows us to apply the methods we developed for the single-demand case and prove the second welfare theorem.

**Lemma 2. (Polytope Lemma)** For every $\mu_i \in Co(B_i)$, the upper contour set $U_i(\mu_i) = \{\mu \in Co(B_i) : V_i(\mu) \geq V_i(\mu_i)\}$ of assignments better than $\mu_i$ for agent $i$ is a convex polytope.

The proof of this lemma is in Appendix B. The key claim of the lemma is that the upper contour set is a polytope. To get a sense for why this claim is true consider the example illustrated in Figure 2. In the figure, agent $i$ has four possible bundles, $B_i = \{b_{i1}, ..., b_{i4}\}$, and the the convex hull $Co(B_i)$ takes the shape of the rhomboid. The highlighted dot represents an expected assignment $\mu_i$. This expected assignment is a convex combination of $\{b_{i1}, b_{i3}, b_{i4}\}$ and it is also a convex combination of $\{b_{i2}, b_{i3}, b_{i4}\}$. Indeed, by the well-known Carathéodory’s theorem, any expected assignment in $Co(B_i)$ is a convex combination of just three extreme points in $B_i$.

\footnote{We thank Jordi Massó for pointing this out.}
unique, and any other representations of $\mu_i$ as convex combinations of $\{b_{i1}, b_{i2}, b_{i3}, b_{i4}\}$ can be decomposed as a convex combinations of these two 3-point convex combinations. Taking into account that $V_i(\mu_i)$ is the maximum of a linear function, to calculate $V_i(\mu_i)$ we only need to know the utility $V$ at these two 3-point convex combinations. This analysis remains valid for any expected assignment in the interior of the triangle span by points $A, b_{i3},$ and $b_{i4}$. Thus, the aforementioned triangle can be divided into a finite number (here: two) of regions on which the set of bundles implementing $\mu$ is convex combinations of these two 3-point convex combinations. Taking a neighborhood of the expected assignment $g$ by a ball, and it also illustrates the parallel linear indifference curves and the direction in which utility increases.

Lemma 2 enables us to leverage the methods we developed in Section 3 to prove Theorem 3. The proof, similarly to the proof of Theorem 1, leverages our general Polytope Separation Lemma (Lemma 1).

**Proof of Theorem 3.** Let $Y = \{m \in \sum_{i \in I} \text{Co}(B_i) : m \leq S\}$ be the set of feasible aggregate expected allocations. Notice that $Y$ is a polytope to which the expected assignment $\mu^*_i = \sum_{i \in I} \beta_i q^*_i$ of $Q^* = \{q^*_i\}_{i \in I}$ belongs. Denote the set of aggregate Pareto-improvements by

$$Z = \left\{ m \in \sum_{i \in I} \text{Co}(B_i) \mid (\exists (\mu_i)_{i \in I}) \left( \sum_{i \in I} \mu_i = m \land (\forall i \in I) V_i(\mu_i) \geq V_i(\mu^*_i) \land (\exists i) V_i(\mu_i) \geq V_i(\mu^*_i) \right) \right\}.$$ 

Because $Q^*$ is not ex-ante Pareto-dominated by any other feasible-in-expectations random assignment, $Z \cap Y = \emptyset$. Furthermore, the aggregate upper contour set $U = \sum_{i \in I} U_i(\mu_i)$ is a closure of $Z$ and, by Lemma 2, $U$ is a polytope.

To be able to apply our Polytope Separation Lemma it remains to verify that for no $z \in Z$ and $y \in Y$, there is $\varepsilon > 0$ such that $y - \varepsilon(z - y) \in U$. By way of contradiction suppose there are such $z, y$ and $\varepsilon$. Then, there is some $\mu = (\mu_i)_{i \in I}$ such that $\sum_{i \in I} \mu_i = y - \varepsilon(z - y)$ and, for all $i \in I$, $V_i(\mu_i) \geq V_i(\mu^*_i)$. Because $z \in Z$ there is $\mu' = (\mu'_i)_{i \in I}$ such that $\sum_{i \in I} \mu'_i = z$ and, for all $i \in I$, $V_i(\mu'_i) \geq V_i(\mu^*_i)$, with strict inequality for some $i$. Consider the expected assignment $\mu'' = \frac{1}{1+\varepsilon} \mu + \frac{\varepsilon}{1+\varepsilon} \mu'$. By construction, $\sum_{i \in I} \mu''_i = y \leq S$, and, by convexity of $V_i$ established in Lemma 2, for all $i \in I$ we have $V_i(\mu''_i) \geq \frac{1}{1+\varepsilon} V_i(\mu_i) + \frac{\varepsilon}{1+\varepsilon} V_i(\mu'_i) \geq V_i(\mu^*_i)$, with strict inequality for some $i$. This contradicts the fact that $Q^*$ is strongly efficient.
Thus we can apply the Polytope Separation Lemma to conclude that there is a hyperplane that fully separates $Y$ and $Z$. The rest of the proof is standard and follows the same step as the analogous part of Theorem 1 above. QED

4.2 First Welfare Theorem

An immediate question is whether all equilibrium outcomes are strongly efficient? We address this question by proving the First Welfare Theorem for strong efficiency under two assumptions. We assume that every agent buys a lowest-cost (cheapest) among all optimal affordable lotteries, a standard assumption in the analysis of the psuedomarkets introduced and motivated by Hylland and Zeckhauser (1979). The lowest-cost assumption is, for instance, implied by the generic assumption that each agent has a unique favorite bundle, which immediately implies that each agent buys a cheapest favorite affordable bundle. We also restrict attention to equilibria satisfying the complementary slackness condition: $p^x > 0$ implies that there is no excess supply of good $x$, $\sum_{i \in I} \beta^x_i q_i^* = |x|$. ²³

Theorem 4. (First Welfare Theorem) Let $Q^*$ be an equilibrium assignment with prices $p^* \in \mathbb{R}^{[X]}_+$ and budgets $(w^*_i)_{i \in I} \in \mathbb{R}^{[I]}_+$ such that the complementary slackness condition is satisfied and each agent buys one of her lowest-cost optimal affordable lotteries over bundles. Then, $Q^*$ is strongly efficient.

Proof. By way of contradiction, suppose $Q^* = \{q_i^*\}_{i \in I}$ is not strongly efficient. Then there is an expected allocation $(\mu_i)_{i \in I}$ such that $\sum_{i \in I} \mu_i \leq S$ and $V_i(\mu_i) \geq q_i^* \cdot v_i$ for all $i \in I$, with at least one inequality strict. If an agent $i$ is not satiated under $q_i^*$—that is with positive probability her outcome is worse than her most preferred bundle—then $p^* \cdot \mu_i \geq p^* \cdot \beta^x_i q_i^*$ by the same argument that works in standard competitive equilibrium theory with non-satiated agents.²⁴ If agent $i$ is satiated then the same inequality holds provided she bought the least expensive most-preferred lottery. The same argument, gives us $p^* \cdot \mu_i > p^* \cdot \beta^x_i q_i^*$ for agents $i$ for whom the inequality $V_i(\mu_i) \geq q_i^* \cdot v_i$ is strict. Summing up the inequalities over agents, we obtain $\sum_{i \in I} p^* \cdot \mu_i > \sum_{i \in I} p^* \cdot \beta^x_i q_i^*$. In particular, there is an object $x$ with positive price $p^x > 0$ and such that $\sum_{i \in I} \mu_i^x > \sum_{i \in I} \beta^x_i q_i^*$. Because $p^x > 0$, the complementary slackness assumption implies that $\sum_{i \in I} \beta^x_i q_i^* = |x|$ (no excess supply). We thus obtain a contradiction with the assumption that $\sum_{i \in I} \mu_i \leq S$. QED

²³The complementary slackness condition was missing in 2017 drafts; other than fixing this omission, there is no substantive difference between the current draft and the 2017 draft.

²⁴Suppose $p^* \cdot \mu_i < p^* \cdot \beta^x_i q_i^*$ and let $b_i$ be a most preferred bundle of agent $i$. We can then find a small weight $\alpha > 0$ such that $V_i(ab_i + (1 - \alpha)\mu_i) > q_i^* \cdot v_i$ and $p^* \cdot (ab_i^* + (1 - \alpha)\mu_i) \leq p^* \cdot \beta^x_i q_i^*$, contradicting that $q_i^*$ was an optimal choice in $i$’s budget set.
4.3 Existence

The final question to answer is whether strongly efficient assignments (and hence competitive equilibria) exist. The potential subtlety here is that strongly efficient feasible random assignment need to be favorable compared to both feasible and unfeasible random assignment of bundles. It turns out that in general a feasible random assignment of bundles that is strongly efficient might not exist.\(^{25}\) However, it does exist when preferences over bundles are strict and since this is a generic property so is the existence of strongly efficient assignments.

**Theorem 5. (Existence)** If preferences over bundles do not show indifferences then a feasible random assignment of bundles that is strongly efficient always exists.

The proof is immediate: under strict preferences any serial dictatorship generates a strongly efficient assignment.

5 Conclusion

We have established the Second Welfare Theorem for the random and multi-unit assignment problem without transfers. Thus, in this setting every efficient assignment can be accomplished by a price mechanism. In addition to the substantive message, we developed a novel approach to analyzing markets in which agents’ preferences are only weakly convex and fail the standard local non-satiation assumption.

A Proof of Lemma 1 (Full Separation Lemma)

We say that \(X\) is fully separated from \(Y\) when \(p \cdot x > w \geq p \cdot y\) for all \(x \in X\) and all \(y \in Y\).

We say that \(Z\) is separated from \(Y\) when \(p \cdot z \geq w \geq p \cdot y\) for all \(z \in Z\) and all \(y \in Y\).

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\(^{25}\)We illustrate it in the following example that builds on the first example and the proposition of this section. Recall that \(B = \{(1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1)\}, S = (1, 1, 1, 1), v_1 = (1, 1-\varepsilon, 0, 0, 1-\varepsilon, 1)\) and \(v_2 = (0, 1-\varepsilon, 1, 1, 1-\varepsilon, 0)\) with \(\varepsilon \in (0, \frac{1}{2})\). The set of feasible deterministic allocations is constituted by \(D_1 = ((1, 1, 0, 0), (0, 0, 1, 1))\), \(D_2 = ((1, 0, 1, 0), (0, 1, 0, 1))\), \(D_3 = ((1, 0, 0, 1), (0, 1, 1, 0))\), \(D_4 = ((0, 1, 1, 0), (1, 0, 0, 1))\), \(D_5 = ((0, 1, 0, 1), (1, 0, 1, 0))\), \(D_6 = ((0, 0, 1, 1), (1, 1, 0, 0))\). None of these deterministic assignments give maximum expected utility 1 to both agents. Every feasible random assignment of bundles is a lottery over \(D = \{D_1, \ldots, D_6\}\). However, all of these deterministic assignments (and thus all of the feasible random assignments of bundles) are dominated by the unfeasible random assignment \(Q = (q_1, q_2)\) where \(q_1 = (1/2, 0, 0, 0, 0, 1/2)\) and \(q_2 = (0, 0, 1/2, 1/2, 0, 0)\) because it gives the maximum expected utility 1 to each agent. In particular, no feasible assignment is strongly feasible. In light of our results, this implies that no feasible assignment is supported as a competitive equilibrium in which every agent buys the cheapest affordable bundle. There are however equilibria—not satisfying the cheapest-bundle assumption—that support some feasible bundles. For instance, the deterministic allocation \(D_1\) might be sustained by prices \(\tilde{p} = (100, 100, 0, 0)\) and budgets \(\tilde{w}_1^* = 200, \tilde{w}_2^* = 0\). In this equilibrium, agent 1 buys a positive price bundle even though this agent could have bought the equally optimal bundle \((0, 0, 1, 1)\) at zero cost. Notice the role of the indifference between favorite bundles in this outcome.
The lemma is easy when $n = 1$. To prove the general case, we will proceed by induction supposing that the lemma is true in dimensions lower than $n \geq 2$.

First notice that $Y \cap Z$ is of dimension lower than $n$. Indeed, if $Y \cap Z$ is of dimension $n$ then there would be an open ball $B \subset Y \cap Z$ around some point $y^* \in Y \cap Z$. Taking any $z \in Z$ and setting $\delta = z - y^*$ we would find an $\epsilon > 0$ and a point $y - \epsilon \delta \in B$ contrary to $y - \epsilon \delta \notin Z$.

$Y \cap Z$ is of dimension lower than $n$, implying that $ri(Y) \cap ri(Z) = \emptyset$, where $ri$ stands for the relative interior of a set. Since $ri(Y)$ and $ri(Z)$ are both convex sets, the standard separating hyperplane theorem implies that there is a hyperplane $H$ that separates $Y$ and $Z$. If $Z$ is disjoint with $H$ or if $Y$ is disjoint with $H$ then the lemma is true. Suppose thus that $H \cap Z$ and $H \cap Y$ are non-empty.

Notice that $Z' = Z \cap H$, $Y' = Y \cap H$, and $\tilde{Z}' = \tilde{Z} \cap H$ satisfy the assumptions of the lemma in the linear space $H$. By the inductive assumption, there is an $n - 2$ dimensional hyperplane $H' \subsetneq H$ fully separating $Z'$ from $Y'$ in $H$, and separating $\tilde{Z}'$ from $Y'$ in $H$. Notice that $H'$ splits $H$ into two open half-spaces. Let $H^Z \subset H$ be the open half-space with empty intersection with $Y'$ (and hence with $Y$) and $H^Y \subset H$ be the open half-space with empty intersection with $\tilde{Z}'$ (and hence with $\tilde{Z}$).

To conclude the proof, look at $n - 1$ dimensional hyperplanes that contain $H'$. Since $\tilde{Z}$ and $Y$ are polytopes, at least one of these hyperplanes, say $H^2 \neq H$, also separates $\tilde{Z}$ and $Y$. Indeed, if only $H$ did so, then either $H^Y$ would need to have a non-empty intersection with $\tilde{Z}$, or $H^{\tilde{Z}}$ would need to have a non-empty intersection with $Y$, a contradiction.

Now, both $H$ and $H^2$ separate the $n$-dimensional space into two open half-spaces. Let $H(\tilde{Z})$ be the half-space bounded by $H$ and disjoint with $Y$, and define analogously $H^2(\tilde{Z})$, $H(Y)$, and $H^2(Y)$. Now $H(\tilde{Z}) \cap H^2(Y)$ is disjoint with both $Y$ and $\tilde{Z}$ and $(H(Y) \cap H^2(\tilde{Z})) \cup (H(\tilde{Z}) \cap H^2(Y))$ contain many hyperplanes. Take any such hyperplane $H^*$. This hyperplane separates $\tilde{Z}$ from $Y$ and it fully separates $Z$ from $Y$. To show this, recall that $H^* \cap H^2 = H^* \cap H = H'$. All of the points in $H^* \setminus H'$ are outside both $Y$ and $\tilde{Z}$. As for the points in $H'$, the induction argument guarantees that none of them belongs either to $Z$ or to $ri(Y)$. QED

### B Proof of Lemma 2 (Polytope Lemma)

The next two lemmas jointly imply the result.

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26 An alternative proof of Lemma 1 can be based on McLennan (2002) Separating Hyperplane Theorem; we would like to thank Andrew McLennan for making this point. McLennan’s theorem can also be used to prove our main result; we provide such an alternative proof of our main result in Appendix B.
Lemma 3. (Convexity) Preferences represented by \( V_i \) are convex.

Proof. Take \( \lambda \in [0, 1] \) and \( \mu_i, \mu'_i \in Co(B_i) \). We need to show that \( \lambda V_i(\mu_i) + (1 - \lambda) V_i(\mu'_i) \leq V_i(\lambda \mu_i + (1 - \lambda) \mu'_i) \). By the definition of \( V \) there is \( q \in \Delta(B_i) \) be such that \( \beta_i q = \mu_i \) and \( V_i(\mu_i) = q \cdot v_i \). Similarly, there is \( q' \in \Delta(B_i) \) such that \( \beta_i q' = \mu'_i \) and \( V_i(\mu'_i) = q' \cdot v_i \). Then,

\[
\begin{align*}
\lambda V_i(\mu_i) + (1 - \lambda) V_i(\mu'_i) &= [\lambda q + (1 - \lambda) q'] \cdot v_i \\
&\leq \max_{\{q'' \in \Delta(B_i) | \beta_i q'' = \lambda \mu_i + (1 - \lambda) \mu'_i \}} q'' \cdot v_i \\
&= V_i(\lambda \mu_i + (1 - \lambda) \mu'_i)
\end{align*}
\]

where the inequality follows because \( \beta_i [\lambda q + (1 - \lambda) q'] = \lambda \mu_i + (1 - \lambda) \mu'_i \), and hence \( q'' = \lambda q + (1 - \lambda) q' \) is in the set the maximum above is taken over. QED

Lemma 4. (Local Affinity) Let \( i \) be an agent. Let \( L \) be the linear space spanned by \( B_i \) and let \( d \) be its dimension. For almost every \( \mu_i \in Co(B_i) \), there exists a convex \( L \)-neighborhood \( M \subseteq Co(B_i) \) of \( \mu_i \) such that \( V_i \) is an affine function of \( \mu \) on \( M \); that is, for all \( \mu, \mu' \in M \) and \( \lambda \in [0, 1] \), \( V_i(\lambda \mu + (1 - \lambda) \mu') = \lambda V_i(\mu) + (1 - \lambda) V_i(\mu') \).

Proof. The set \( D \) of expected assignments in \( Co(B_i) \) that can be represented as a convex combination of \( d \) or fewer points in \( B_i \) is of measure zero in \( L \). This claim follows from two observations. First, the convex hull of any \( d \) or fewer points is of dimension at most \( d - 1 \), and hence of measure zero in the \( d \)-dimensional space \( L \). Second, there is only a finite number of subsets in \( B_i \) because \( B_i \) itself is finite.

Let us fix an expected assignment \( \mu_i \in Co(B_i) - D \). Let \( B_i(\mu_i) \) be the set of all \( B \subseteq B_i \) such that \( \|B\| \leq d + 1 \) and \( \mu_i \) is a convex combination of elements from \( B \). Because \( \mu_i \not\in D \) we infer that each \( B \in B_i(\mu_i) \) has exactly \( d + 1 \) elements. \( B_i(\mu_i) \) is finite because \( B_i \) is finite. \( B_i(\mu_i) \) is nonempty because Carathéodory’s Theorem tells us that \( \mu_i \) can be represented as a convex combination of \( d + 1 \) elements of \( B_i \).\(^{27}\) Furthermore, for any \( B \in B_i(\mu_i) \) there is exactly one convex combination of elements of \( B \) that gives \( \mu_i \). Indeed, if there were two such convex combinations then \( \mu_i \) would also be a convex combination of elements from a proper subset of \( B \); a contradiction because \( \|B\| = d + 1 \) and \( \mu_i \not\in D \).

By definition of \( V_i \), there is \( B \in B_i(\mu_i) \) such that \( V_i(\mu_i) = q \cdot v_i \) for some \( q \in \Delta(B_i) \) such that \( \mu_i = \beta_i q_i \), and \( q_b > 0 \) iff \( b \in B \). Let us denote by \( \mu^1, ..., \mu^{d+1} \) the expected assignments that belong to \( B \). For any \( \varepsilon \in (0, \min \{q^b | b \in B\} \cup \{1 - q^b | b \in B\}) \), the set \( B^\varepsilon \) of convex

\(^{27}\)We thank Jordi Massó for directing us to Carathéodory’s Theorem.
combinations of elements of $B$ with weight on each $b \in B$ taken from $(q^b - \varepsilon, q^b + \varepsilon)$ is a convex full-dimensional open subset of $Co(B_i)$, and hence a convex $L$-neighborhood of $\mu_i$.

We claim that for sufficiently small $\varepsilon > 0$, all expected assignments in $B^\varepsilon$ have a unique decomposition as a convex combination over a subset of $B_i(\mu_i)$, and this unique decomposition is over $B$. Indeed, if not then there is a sequence of $\mu_i^\ell \in Co(B_i)$ that tends to $\mu_i$ as $\ell \to \infty$ and such that all $\mu_i^\ell$ have at least two convex decompositions over subsets of $B_i(\mu_i)$. Same argument as above shows that then all $\mu_i^\ell \in D$ and we can select a subsequence $\ell_n$ such that all $\mu_i^{\ell_n}$ are convex combinations of the same $d$ (or fewer) points in $B$. But then $\mu_i = \lim_{n \to \infty} \mu_i^{\ell_n}$ would also be a convex combination of the same $d$ (or fewer) points in $B$; a contradiction.

Take $\varepsilon$ that is sufficiently small in the sense of the above claim. Then, $\mu_i$ is an arbitrary element of the full measure subset of $Co(B_i)$, and the uniqueness of the convex decomposition implies that for all $q \in \Delta(B_i)$ such that $q^b > 0$ iff $b \in B$ and $\beta_i q$ belongs to the convex neighborhood $M = B^\varepsilon$ of $\mu_i$, the utility $V_i(\beta_i q) = q \cdot v_i$. Thus, $V_i$ is affine on $M$. QED

References


