

The First-Price Principle of Maximizing Economic Objectives*

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August 2023

Abstract

Allowing for any restrictions on bid spaces, we show that the first-price auction format is sufficiently flexible to achieve in equilibrium any Bayesian-constrained standard objective, including maximizing or minimizing revenue, welfare, bidder surplus, and Gini mean difference as well as linear combinations of them. This first-price principle allows us to analyze problems that are beyond the scope of Myersonian mechanism design. Our further results simplify the analysis of first price auctions with arbitrary bid spaces and establish the existence of monotonic pure-strategy equilibria in these auctions.

1 Introduction

First-price auctions have been used to sell objects throughout history. While it is known that the first-price auction format can be used to achieve objectives such as welfare maximization or revenue maximization, the auctioneer may be interested in other types of objectives. For instance, FTC optimizes the resulting consumer surplus. An auctioneer with distributional concerns may want to allocate the objects in a way that minimizes inequality among bidders with different valuations, or that balances inequality measures and revenue maximization.

*First posted draft May 18 2015; first presentation June 2, 2015. We would like to thank Ivan Balbuzanov, Simon Board, Piotr Dworzak, Alia Gizatulina, Simon Loertscher, Moritz Meyer-ter-Vehn, Afshin Nikzad, Ichiro Obara, Ludvig Sinander, William Zame, audiences at UCLA, SAET 2021, GAMES 2021, Swiss Theory Day, SSES, and the UCLA students who took the Designed Markets class in 2014.

 This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program grant agreement No 866376.

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In this paper, we show that a large class of objectives, that includes the above examples, can be maximized by using first-price auctions. This insight relies on the objective satisfies three standard conditions: anonymity, the convexity of the maximand set, and quasi-convexity.

By first price auction we mean any mechanisms in which bidders submit bids from a set of allowed bids, the object is allocated to one of the highest bidders, the winning bidder pays their bid, while the remaining bidders pay nothing. It is sufficient for our result to consider symmetric auctions in which each bidder chooses from the same bid space and has the same chance of winning an object in tie-breaking. The flexibility of the first-price auctions we study is caused by us allowing arbitrary restrictions on the set of allowed bids. A reserve price is an example of such a restriction but we show that sometimes optimality might require imposing further restrictions such as, for instance, a (binding) maximum price.

Our main theorem shows that any standard objective can be achieved by a first-price auction. We prove this “first-price principle” in a general private values environment in which multiple buyers participate in a mechanism to win an object. Our environment incorporates as a special case the standard environment of Myerson (1981) and relaxes two of Myerson’s assumptions. We allow type distributions with arbitrary supports, finite or infinite, as opposed to requiring the support to be an interval; we also allow arbitrary type distributions and not only Lebesgue continuous ones. By showing that restricting attention to first-price auctions is sufficient for optimal design, we make all such design problems tractable.

We derive our main theorem from two steps. In the first step we restrict attention to finite type spaces, and show that any symmetric mechanism is a lottery over first-price auctions. This auxiliary result can be restated as showing that the set of extreme points of symmetric mechanisms consist of first-price auctions.¹ Our analysis of discrete type spaces blends in a new way Border’s (1991) analysis of interim allocations and Manelli and Vincent (2010) and Gershkov et al.’s (2013) approach to Bayesian-Dominant-strategy allocation equivalence. In the second step, we treat the general type space as a limit case of discrete type spaces and show that the validity of the first-price principle for discrete type spaces is preserved in the limit.

Our focus on the first-price principle is at the source of our ability to relax the Myersonian assumptions. Unlike Myerson-inspired mechanism design analysis, we do not use incentive compatibility to pin the slope of participants’ mappings from types to payoffs; it is this slope determination that restricts Myersonian analysis to Lebesgue continuous distributions with interval supports.

As to the best of knowledge, our work is the first to study first price auctions with general

¹In finite types space, we additionally show that lotteries over first-price auctions can achieve any anonymous objective whose maximand set is convex.

bid spaces, we also provide additional results whose aim is to facilitate the analysis of such auctions.² In particular, we prove the existence of monotonic pure-strategy equilibria for first-price auctions with compact bid spaces. We also show that any equilibrium in a first-price auction with any bid space remains an equilibrium if we replace the bid space by its closure. Thus, from design perspective, the compactness assumption in our existence results is without loss of generality when the type space is bounded.

Our focus on the first price auction is not merely driven by the canonical nature of this mechanism class: we show that an analogue of our main result fails if we replace first-price auctions with second-price auctions. There are environments in which even such standard objectives as revenue maximization cannot be achieved by symmetric second price auctions.

Our results contribute to the literature creating new tools for mechanism design. Most related is Kleiner, Moldovanu, and Strack (2021). They focus on characterizing optimal allocations while we characterize the space of optimal mechanisms. While they maintain the standard Myersonian continuity and support assumptions, our approach allows us to relax them. In an earlier work, Carbajal and Ely (2013) maintain the continuity assumptions but relax the indivisibility assumption. In a later work, Nikzad (2022) maintains the continuity assumptions but allows additional constraints on the design problem.

Earlier works on mechanism design with discrete type spaces focused on extending the slope determination behind revenue equivalence, and explored its limits, e.g. Lovejoy (2006). Beyond discrete space, the limits of the Myersonian slope approach was explored in Che and Gale (2006), who showed that first-price auctions yield higher expected revenue when risk-averse bidders are financially constrained, and in Chung and Olszewski (2007) and Heydenreich et al. (2009), who characterized the restrictions on the primitives that lead to the revenue equivalence result. Our work takes a different approach by allowing more restrictions on the bid space and shows that the first-price format dominates the second-price auction format in terms of the expected revenue.

By allowing a wide range of objective functions, our results are related to Condorelli (2013)—who shows that maximizing a linear combination of assignment values leads to assigning objects to agents with the highest willingness to pay—and Akbarpour, Dworzak, and Kominers (2020)—who shows that mechanisms maximizing a linear combination of revenue and welfare combine assortative (or market) allocation and uniform randomization.

²Specific restrictions have been studied, primarily the above-mentioned reserve prices (cf. J. G. Riley and W. F. Samuelson (1981) and Myerson (1981)). Of relevance is also the literature on bidders' budget constraints in first-price auctions (cf. Che and Gale (1998b), Pai and R. Vohra (2014)); the main differences with this literature is that we allow the designer to optimize the bid space and that we allow much more class of bid spaces. Beyond first-price auctions, there is also work on bid caps in all pay auctions, cf. Che and Gale (1998a), Szech (2015), Olszewski and Siegel (2019).

The objectives they study are standard in our sense and hence our results are applicable to their environment. Ours and their work rely on different methodologies: we establish our results for finite type spaces (in which revenue equivalence might fail) and extend them for more general type spaces, while the above three papers study continuum type spaces (in which revenue equivalence obtains).

Our existence result complement other papers establishing existence of equilibria in first-price auctions with unconstrained (or constrained only by the reserve price) bid spaces, e.g. J. G. Riley and W. F. Samuelson (1981). Philip J. Reny (2011) proved existence of monotone pure-strategy equilibria in finite Bayesian games. His result implies the existence of equilibria in our auctions when the bid space is finite and we build on his existence result to prove existence for more general bid spaces.³ Our closure lemma has no analogues we know of.

2 Model

We study an environment with n bidders indexed by $i \in N = \{1, \dots, n\}$. Each bidder i has a utility type $\theta_i \in \Theta$, where Θ is a compact subset of \mathbb{R} . The discrete case $\Theta = \{\theta^1, \dots, \theta^T\}$ will play a special role in our proofs. We assume that each bidder's types are distributed according to distribution π and that the types are distributed i.i.d. across bidders. For brevity we also use π to refer to the joint distribution, and—in the discrete case—we write $\pi(\theta_1, \dots, \theta_n) = \pi(\theta_1) \dots \pi(\theta_n)$. In the discrete case, we index the types so that θ^t is strictly increasing in t .⁴

We study the allocation of a single indivisible object. A pure outcome is a vector $\mathbf{x} = (\mathbf{a}, \mathbf{p})$, where $\mathbf{a} = (a_1, \dots, a_n) \in \{0, 1\}^n$ such that $\sum_{i \in N} a_i = 1$ is the vector of allocations and $\mathbf{p} = (p_1, \dots, p_n) \in [\underline{p}, \infty)^n$ is the vector of payments for some $\underline{p} \leq 0$.⁵ When $a_i = 1$ and $p_i = p$ we say that agent i receives the object and pays p . The space of pure outcomes is \mathbf{X} ;

³Our extension builds on the ideas from Philip J. Reny and Zamir (2004), who show the existence of monotone pure-strategy equilibria in first-price auctions for standard bid space with bidder-specific reserve prices. They first show the existence of an equilibrium when bid space is finite and there is no tie-breaking which can be ensured by not allowing common bids for different bidders, and show that the limit of the equilibria via approximation is still an equilibrium. We also employ a limit argument made more complex because we need to take care of exogenous tie-breaking. For other related limit arguments, see, e.g., Woodward (2016). Olszewski, Philip J Reny, and Siegel (2023) analyze the existence of equilibria in first price auctions with asymmetric type distributions.

⁴The strict monotonicity is without loss because we can represent the case when two types are equal, $\theta^t = \theta^{t+1}$, by using only one of these types and assigning it probability $\pi(\theta^t) + \pi(\theta^{t+1})$ for each profile of other bidders' types. In order to simplify exposition, when using superscripts (and subscripts) as above, we suppose that $\pi(\theta^{T+1})$ and similar undefined expressions take value 0.

⁵Bounding the possible payments from below means that we bound the potential subsidies the seller might offer to the bidders. We may interpret this bound as reflecting the seller participation or budget constraint. Imposing this is not needed when we study objectives such as revenue-maximization but it simplifies our analysis of more general objectives by ensuring the space of relevant mechanisms is compact.

an individual outcome (a_i, p_i) of each bidder i belongs to $X = \{0, 1\} \times \mathbb{R}$. An outcome is a lottery over pure outcomes. Given a pure outcome $\mathbf{x} = (\mathbf{a}, \mathbf{p})$, the utility of each bidder i is given by $a_i \theta_i - p_i$; an agent utility from an outcome is the resulting expected utility. A social rule is a mapping from the space of type profiles Θ^N to the space of outcomes $\Delta(\mathbf{X})$. Given a social rule \mathbf{y} and a profile of types θ , we denote by $\mathbf{q}(\theta) = (q_i(\theta))_{i \in N}$ the vector of resulting expected allocations and by $\mathbf{t}(\theta) = (t_i(\theta))$ the vector of resulting expected payments, where each $q_i(\theta)$ is the expected value of a_i and $t_i(\theta)$ is the expected value of p_i calculated using the lottery $\mathbf{y}(\theta)$

Bidders participate in a mechanism that allocates a single indivisible object and collects payments from the bidders. A mechanism is given by the space of actions and a choice function. The space of actions is a set \mathcal{B} ; because auctions play a focal role in our analysis, we refer to actions $b \in \mathcal{B}$ as bids and to \mathcal{B} as the bid space. For the simplicity of exposition, we assume that \mathcal{B} is the same for all bidders; an assumption that is without loss of generality in the study of Bayesian Nash equilibria.⁶ A outcome function \mathbf{f} maps any profile of bids $\mathbf{b} = (b_i)_{i \in N}$ into a lottery $\mathbf{f}(\mathbf{b})$ over the space of pure outcomes \mathbf{X} ; using standard notation, we denote the space of the resulting lotteries by $\Delta(\mathbf{X})$. We say that a mechanism generates a social rule if there exists an equilibrium $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ such that $\mathbf{f}(\boldsymbol{\sigma}(\theta))$ is the same as the social rule applied to type profile θ for every profile in Θ^n . A social rule is implementable if there exists a mechanism that generates it and is individually rational if the expected utility of each bidder is weakly greater than 0 at any outcome in the range of the social rule.

A canonical example of a mechanism is a direct mechanism, in which $\mathcal{B} = \Theta$. We are also interested in auctions. An auction is a mechanism such that \mathcal{B} is a subset of $[\underline{p}, \infty) \cup \{\emptyset\}$ and $\emptyset \in \mathcal{B}$, and the outcome function \mathbf{f} has the property that if at least one bidder submitted a bid in \mathbb{R} (a real bid) then for exactly one bidder i whose bid is weakly higher than all other submitted real bids $a_i = 1$, and for all other bidders $j \neq i$ the allocation $a_j = 0$; otherwise, for all bidders $a_i = 0$. A first-price auction is an auction such that $p_i = b_i$ if $a_i = 1$ and \underline{p} otherwise, for any $\mathbf{x} \in \text{supp}(\mathbf{f}(\mathbf{b}))$ and $\mathbf{b} \in \mathcal{B}$. A second-price auction is an auction such that, for any $\mathbf{x} \in \text{supp}(\mathbf{f}(\mathbf{b}))$ and for any $\mathbf{b} \in \mathcal{B}^n$, if $a_i = 1$ then

$$p_i = \begin{cases} \min \mathcal{B} \cap \mathbb{R} & \text{if } i \text{ is the only bidder submitting a real bid} \\ \max\{b_j | j \in N - \{i\} \text{ and } b_j \neq \emptyset\} & \text{if some bidder } j \neq i \text{ submitted a real bid} \end{cases}$$

and if $a_i = 0$ then $p_i = 0$. Note that $\min \mathcal{B} \cap \mathbb{R}$ is usually called the reserve price. Note also

⁶Indeed, for any equilibrium in any mechanism in which each agent i has a space \mathcal{B}_i , we can find an outcome function and strategies in a mechanism in which each agent has the space $\mathcal{B} = \cup_{i \in N} \mathcal{B}_i$ such that the strategies are in equilibrium and the mechanism with action spaces all being \mathcal{B} implements the same mapping from types to outcomes as the original mechanism with action spaces \mathcal{B}_i .

that in both first-price and second price auction, we allow the design of the bid space that goes beyond imposing a reserve price; in examples below we show that this additional degree of design freedom is necessary to achieve standard objectives.

Fix a social rule $\mathbf{y} = (y_1, \dots, y_n)$. Each bidder i with a type θ_i can compute the interim expected outcome (Q_i, T_i) by $\sum_{\theta_{-i} \in \Theta_{-i}} y_i(\theta_i, \theta_{-i}) \pi(\theta_i, \theta_{-i})$ and we say that Q_i is the interim expected allocation and T_i is the interim expected payment. We call $\mathbf{Q} = (Q_i)_{i \in N}$ the interim allocation rule and $\mathbf{T} = (T_i)_{i \in N}$ the interim payment rule. We call (\mathbf{Q}, \mathbf{T}) a reduced-form of \mathbf{y} and we also say that a mechanism $(\mathcal{B}, \mathbf{f})$ implements a reduced form mechanism (\mathbf{Q}, \mathbf{T}) if it generates a social rule \mathbf{y} such that (\mathbf{Q}, \mathbf{T}) is a reduced form of \mathbf{y} . Whenever we refer to a reduced-form mechanism, we always assume that there exists a mechanism $(\mathcal{B}, \mathbf{f})$ that implements it. We then say that an interim allocation rule \mathbf{Q} is feasible if there exists a reduced form mechanism (\mathbf{Q}, \mathbf{T}) and, a priori, we allow interim allocation rules that are not necessarily feasible. A reduced form mechanism (\mathbf{Q}, \mathbf{T}) is incentive compatible if $Q_i(\theta)\theta - T_i(\theta) \geq Q_i(\hat{\theta})\theta - T_i(\hat{\theta})$ for all $i \in N$ and $\theta, \hat{\theta} \in \Theta$; notice that the incentive compatibility requires that agents are willing to report their true types to the reduced form mechanism; in this sense the reduced form mechanism is a direct reduced form mechanism. A reduced form mechanism (\mathbf{Q}, \mathbf{T}) is individually rational if $Q_i(\theta)\theta - T_i(\theta) \geq 0$ for all $i \in N$ and $\theta, x \in \Theta$. An interim allocation rule \mathbf{Q} is symmetric if $Q_i(\theta) = Q_j(\theta)$ for all $i, j \in N$ and $\theta \in \Theta$. A reduced form mechanism (\mathbf{Q}, \mathbf{T}) is symmetric if \mathbf{Q} is symmetric and $T_i(\theta) = T_j(\theta)$ for all $i, j \in N$ and $\theta \in \Theta$. A social rule \mathbf{y} is called symmetric if $y_i(\boldsymbol{\theta}) = y_1(\tau^i(\boldsymbol{\theta}))$ for all i and $\boldsymbol{\theta}$ and transposition τ^i that interchanges the first and i -th coordinates of $\boldsymbol{\theta}$.

3 Objectives

The goal of the allocation is to maximize an objective such as welfare and revenue maximization or inequality minimization. Given π , an objective U_π is a mapping from the set of social rules to \mathbb{R} . We restrict attention to upper semicontinuous mappings so as to ensure that there is an implementable and individual rational social rule that maximizes the objective; note that the set of such rules is compact. We say that a social rule maximizes the objective if it is implementable, individually rational, and there is no other implementable and individually rational social rule that maps to a higher value. We say that a mechanism $(\mathcal{B}, \mathbf{f})$ maximizes the objective if it generates a social that maximizes the objective.

We say that an objective is standard if it satisfies the following five properties:⁷

(i) Anonymity: If \mathbf{y} maximizes the objective then so does $\tau^{-1} \circ \mathbf{y} \circ \tau$ for any bijection $\tau : N \rightarrow N$.

(ii) Convexity of the maximand set: If \mathbf{y}_1 and \mathbf{y}_2 maximize the objective then so does $\frac{1}{2}\mathbf{y}_1 + \frac{1}{2}\mathbf{y}_2$.

(iii) Quasi-convexity: For any symmetric \mathbf{y}_1 and \mathbf{y}_2 , if $\lambda\mathbf{y}_1 + (1 - \lambda)\mathbf{y}_2$ maximizes the objective for some $\lambda \in (0, 1)$, then either \mathbf{y}_1 or \mathbf{y}_2 (or both) maximize the objective.

(iv) Continuity: $U_\pi(\mathbf{y})$ is continuous in \mathbf{y} (in $L^1(\pi)$ metric on \mathbf{y}) and continuous in π (in Prokhorov metric on π).

(v) Revenue-monotonicity: $U_\pi(\mathbf{y})$ is increasing in $\sum_i t_i(\theta)$ for fixed $(q_i(\theta))_{i \in N}$.

Remark 1. The continuity and the revenue-monotonicity assumptions play a role in our analysis of general type spaces. In the special case of finite Θ , this assumption however can be completely relaxed. The revenue-monotonicity constraint can be relaxed if the distribution over types is atom-less. These additional restrictions beyond (i)-(iii) are not needed at any point of our analysis of the finite case.

In addition to the expected welfare and revenue maximization, the class of standard objectives includes seller's expected profit-maximization for any cost function of the seller, as well as linear combinations of expected welfare, revenue, and profits. Other examples of standard objectives include maximizing or minimizing the sum of payoffs of all bidders or of bidders of a specific type or the probability of that bidders with intermediate values win. Maximizing the weighted sum of payments, where weights depend on the bidders' types, is also a standard objective. In contrast, maximizing the variance of payments is an example of an objective that—in general—is not standard in the above sense because it violates our condition (ii).

Another interesting class of standard objectives are objectives minimizing inequality such as the Gini coefficient across bidders and their types

$$\text{Gini}_{BT} = \frac{\sum_{i,j} \sum_{\theta,\theta'} |Q_i(\theta) - Q_j(\theta')| \pi(\theta)\pi(\theta')}{\sum_{i,j} \sum_{\theta} Q_i(\theta)\pi(\theta)},$$

⁷See Bergson (1938) and P. A. Samuelson (1956) for the convexity assumptions; see May (1952), Sen (1970), and Pycia (2019) for anonymity. Because of upper semicontinuity, property (ii) implies that if \mathbf{y}_1 and \mathbf{y}_2 maximize the objective then so does $\lambda\mathbf{y}_1 + (1 - \lambda)\mathbf{y}_2$ for any $\lambda \in [0, 1]$. See Condorelli (2013) and Akbarpour, Dworzak, and Kominers (2020) for a case to go in our analysis beyond the revenue and standard welfare. The former paper study the maximization of a linear combination of type-dependent values the designer puts on assignments and the latter the maximization of type-dependent combination of revenue and welfare; both these objectives are standard in our sense.

as well as the ex ante Gini mean difference across bidders

$$\text{GMD}_B = \sum_{i,j} \sum_{\theta, \theta' \in \Theta} |Q_i(\theta) - Q_j(\theta')| \pi(\theta) \pi(\theta').$$

These objectives satisfy conditions (ii) and (iii) because they can be re-expressed as twice the sums over types $\theta > \theta'$; because the incentive compatibility implies the monotonicity of payoffs $U_i(\theta)$ in types θ , after such re-expression we can drop the absolute values in the formulas. Furthermore, maximizing any linear combinations (including with negative weights) of revenue, welfare, and the ex ante Gini mean difference is also standard.

4 The First-Price Principle

Our main result shows that first-price auctions are sufficient to implement all standard objectives.

Theorem 1. For any standard objective, there exists a bid space \mathcal{B} such that a pure-strategies equilibrium in the first-price auction with bid space \mathcal{B} maximizes the objective.

This result implies that, for standard objectives, the allocation is piecewise assortative, and it is uniformly random on bunched intervals, that is monotonic partitions.⁸

Corollary 1. For all standard objectives, there is a monotone partition of the type space such that the object is assigned to an agent in the highest element of the partition. Furthermore, the elements of the partition are of two types: for some elements the object is assigned uniformly at random to all bidders with types in the element; for other elements the object is assigned uniformly at random to bidders with the highest type.

In particular, this corollary contains the main insight of Akbarpour, Dworzak, and Kominers (2020). In addition, it implies similar insights for other standard objectives, e.g. for the maximization of any linear combination of revenue, welfare, and the ex ante Gini mean difference.

A key step in our analysis shows that any symmetric reduced form mechanism that is incentive compatible and individually rational is equivalent to a lottery over first-price auctions and any such lottery is a symmetric mechanism (we exchangeably use the terms a lottery over mechanisms and a convex combination of the mechanisms). Establishing the decomposition insight for symmetric mechanism is sufficient to prove our opening claim because of the following:

⁸Monotonic partitions are defined in Section 6.

Lemma 1. Any standard objective is maximized by a symmetric social rule.

Proof. Given a standard objective, suppose that a social rule \mathbf{y} maximizes the objective and let \mathcal{T} be the space of all permutations over the set of agents, N . Then the social rule $\frac{1}{|N|!} \sum_{\tau \in \mathcal{T}} \tau^{-1} \circ \mathbf{y} \circ \tau$ is symmetric, individually rational, and implementable. Furthermore, the properties (i) and (ii) of standard objectives ensure that this symmetric social rule maximizes the objective. \square

This lemma extends the straightforward observation that efficient allocations are symmetric and the classical insight that the allocations of revenue-maximizing mechanisms are symmetric as well. The latter insight goes back to Myerson (1981).⁹

We prove Theorem 1 by first establishing it for discrete type spaces and then showing that the result remains true for general type space by approximating them as a limit of finite type spaces.¹⁰

For finite type spaces, we show

Theorem 2. For finite type spaces and any anonymous objective with convex maximand set, there is a lottery over first-price auctions that this objective is maximized in a pure strategy equilibrium of the resulting mechanism.

Note that the designer maximizing a standard objective can without loss restrict the bid space to bids chosen on equilibrium path. Furthermore, when a pure strategy equilibrium is played by bidders with finite type space, then a finite subset of bid space is used on equilibrium path. Hence, if we restrict attention to pure strategy equilibria then considering finite bid spaces is sufficient.¹¹

We prove Theorem 2 for finite type spaces, building on the Theorem 3 below. Theorem 2 and Lemma 1 imply a Theorem 1 for finite type spaces (with no need to rely quasi-convexity of the objective). We then extend Theorem 1 from finite to any general type space by approximating the type space with finite type spaces. We use Helly's selection theorem to select a convergent sequence of first-price auctions that are optimal a sequence of finite type spaces and we show that the limit of this sequence is a first-price auction that maximizes the objective.

⁹Cf. also Lovejoy (2006) and Bergemann and Pesendorfer (2007) who proved this symmetry insight in the context of discrete type spaces.

¹⁰This approach to mechanism design is unusual but has some forerunners; see Philip J. Reny (1999), Philip J. Reny (2011), and Woodward (2016) for the analysis of discontinuous games.

¹¹A similar claim for mixed strategies relies on a more subtle argument and we omit it because we are not relying on it.

Theorem 3. If the type space is finite, every symmetric, incentive compatible, and individually rational reduced-form mechanism (\mathbf{Q}, \mathbf{T}) is a reduced-form of a lottery over equilibria of first-price auctions.

In the theorem we could replace a lottery over equilibria with an equilibrium of the mechanism in which bidders submit bids conditional on the first-price auction drawn by the lottery.¹² We prove Theorem 3 in Section 6 by analyzing the extreme points in the space of all mechanisms (see Theorem 6); in particular we show there that first-price auctions are such extreme points (Theorem 5).

Proof of Theorems 2 and 1 for finite type spaces. By Lemma 1, we can restrict our attention to symmetric mechanisms. Note that the space of implementable and individually rational social rules is convex and compact. Theorem 2 then follows directly from Theorem 3. To establish Theorem 1 notice further that, by the quasi-convexity of the standard objective, the standard objective is maximized by a mechanism $(\mathcal{B}^*, \mathbf{f}^*)$ that generates \mathbf{y}^* which is an extreme point of the space. Note that the set \mathbf{S} of symmetric, incentive compatible, and individually rational reduced form mechanisms is convex and compact. Denote the reduced form mechanism of \mathbf{y}^* by $(\mathbf{Q}^*, \mathbf{T}^*)$. Then, $(\mathbf{Q}^*, \mathbf{T}^*)$ is also at an extreme point of the set \mathbf{S} . By Theorem 3, any extreme points of \mathbf{S} is a reduced form of a first-price auction, so is $(\mathbf{Q}^*, \mathbf{T}^*)$. \square

4.1 Equilibria of First Price Auctions

The existence of equilibria in First Price Auctions with standard bid space (which allow all bids above some reserve price) have been proven by J. G. Riley and W. F. Samuelson (1981). Does their existence remain valid for other bid spaces? For atom-less type spaces we show that equilibria exist provided bid spaces are compact; as the spaces of relevant bids is always bounded, compactness can be weakened to the bid space being topologically closed. Furthermore, it is easy to construct examples in which the pure-strategy equilibrium fails to exist because the bid space is not closed.

Theorem 4. If the type distribution is atom-less and the bid space is compact, there exists a monotone pure-strategy symmetric equilibrium in the first-price auction.

If the bid space is finite, the existence of a pure-strategy symmetric equilibrium is implied by Philip J. Reny (2011). Appendix A provides the proof for the general case in which we

¹²Such conditioning is natural in our analysis because we allow different bid spaces in different first-price auctions.

use the equilibria in the finite bid spaces to construct an equilibrium for the general case.¹³

The theorem restricts attention to compact bid spaces and the following complementary lemma shows that this restriction is without loss of generality in our analysis of First Price Auctions.

Lemma 2. If a symmetric profile of bidding strategies $b(\cdot)$ is an equilibrium in the First Price Auction with bid space \mathcal{B} , then the same profile of strategies is in equilibrium in the First Price Auction with bid space $cl(\mathcal{B})$, where $cl(\mathcal{B})$ is the closure of \mathcal{B} .

The converse does not necessarily hold true. One can construct a type space and \mathcal{B} for which there is an symmetric pure-strategy equilibrium in the First Price Auction with bid space $cl(\mathcal{B})$ but no such equilibrium in the first -price auction with bid space \mathcal{B} .

4.2 Examples Illustrating Theorem 1

We show in two examples how to use the above results to solve for optimal mechanisms in environments in which using the standard Myersonian technique is not applicable. The first example entails a non-Myersonian objective and its solution entails ironing while the second maximizes standard expected revenue but the type distribution fails the assumptions on which the Myersonian approach is built.

Example 1. [Balancing Revenue and Equality]

Suppose there are two bidders and each bidder has a type independently drawn from a uniform distribution on $[0, 1]$. A designer balances maximizing expected revenue and minimizing the disparities in the winning probabilities among different types: the objective U is a mix of the expected revenue and the total variation in the winning probability among different types, $U(\mathbf{Q}, \mathbf{T}) = E [T_1(\theta_1) + T_2(\theta_2)] - \alpha TV (Q_1(\theta_1) + Q_2(\theta_2))$. The parameter $\alpha > 0$ indicates the level of concern about the differences in winning probabilities; the higher α is the more designer is focused on minimizing these disparities and less on revenue. Because the total variation of a monotonic function is simply the difference between its values at the start and end points, this objective is standard and even linear on the set

¹³Allowing for risk aversion and interdependent values, Philip J. Reny and Zamir (2004) established the existence of a monotone pure-strategy equilibrium for finite bid spaces, and by taking the limit, for standard bid space with bidder-specific reserve prices. Our proof also starts with finite bid spaces. By taking the limit we then extend the existence to all compact bid spaces; in the process we need to handle the additional subtleties arising from the nonconvexity of the bid space. Cf. also the classic analysis of J. G. Riley and W. F. Samuelson (1981) who characterized the pure-strategy symmetric equilibrium under standard bid space with a reserve price, as well as Woodward's (2016) limit analysis of finite type spaces in multi-unit auctions.

of incentive-compatible mechanisms. Thus, the objective can be expressed as $U(Q, T) = 2E[T(\theta)] - 2\alpha[Q(1) - Q(0)]$.

Theorem 1 enables us to find an optimal mechanism by looking only at first-price auctions with a pure-strategy equilibrium. We first show that it is sufficient to look at interval bid spaces.

By Lemma 2, it is enough to consider compact bid spaces. In Appendix we show the following.

Claim. Consider a compact bid space $\hat{\mathcal{B}}$ and a corresponding equilibrium bidding function $\hat{b}(\cdot)$ such that $r = \min \hat{\mathcal{B}} > 0$ and $\bar{b} = \max \hat{\mathcal{B}}$. The expected equilibrium value of U given this bid space and the equilibrium is weakly lower than then expected equilibrium value of U given the bid space $\mathcal{B} = [r, \bar{b}]$ and $b(\cdot) = \begin{cases} \frac{1}{2}\theta + \frac{1}{2\theta}r^2 & \text{if } r \leq \theta < \frac{\bar{b}-r^2}{1-\bar{b}} \\ \bar{b} & \text{if } \theta \geq \frac{\bar{b}-r^2}{1-\bar{b}} \end{cases}$.

This claim characterizes the equilibrium strategy $b(\cdot)$ and the designer's problem boils down to maximizing a function with two variables \bar{b} and r , which is:

$$U(\bar{b}, r) = -\frac{1}{3}r^3 + \frac{(\bar{b} - r^2)^3}{3(1 - \bar{b})^3} + r^2 \left(-r + \frac{(\bar{b} - r^2)}{1 - \bar{b}} \right) + \bar{b} \left(1 - \frac{(\bar{b} - r^2)^2}{(1 - \bar{b})^2} \right) - \alpha \left(\frac{1 + (\bar{b} - r)^2}{2(1 - \bar{b})} \right).$$

Figure 1 illustrates the solution for varying values of α .

Example 2 (Type Distribution with an Atom).

Suppose there are two bidders. Each bidder's type is either 0.5 with a probability $p < 1$ or is drawn from a uniform distribution on $[0, 1]$ with the remaining probability $1 - p$, independently. Suppose the designer's goal is to maximize the total surplus.¹⁴ This objective can be achieved through a first-price auction with the bid space $\mathcal{B} = [0, \frac{1}{4}] \cup \{\frac{1+p}{4}\} \cup [\frac{3p+1}{4p+4}, \frac{1}{2}]$. In the first-price auction with the bid space \mathcal{B} , the symmetric equilibrium bidding strategy is

$$b(\theta) = \begin{cases} \frac{1}{2}\theta & \text{if } \theta < \frac{1}{2}, \\ \frac{1+p}{4} & \text{if } \theta = \frac{1}{2}, \\ \frac{p+(1-p)\theta^2}{2(p+(1-p)\theta)} & \text{if } \theta > \frac{1}{2}. \end{cases}$$

¹⁴Were revenue maximization the designer's goal, the optimal bid space would be $\mathcal{B} = [\frac{1}{2}, 1]$ for all atom probabilities p . The reason is that in this example the atom is at the optimal reserve price of the standard case with $p = 0$. If the atom is at value $a < .5$ then the optimal solution has bid space either $[\frac{1}{2}, 1]$ or $[a, 1]$ depending on the values of a and p . If the atom is at value $a > .5$ then the optimal bid space resembles the efficient bid space we construct below and, in general, take the form $[\frac{1}{2}, \lim_{\theta \rightarrow a-} b(\theta)] \cup \{b(a)\} \cup (\lim_{\theta \rightarrow a+} b(\theta), 1]$.

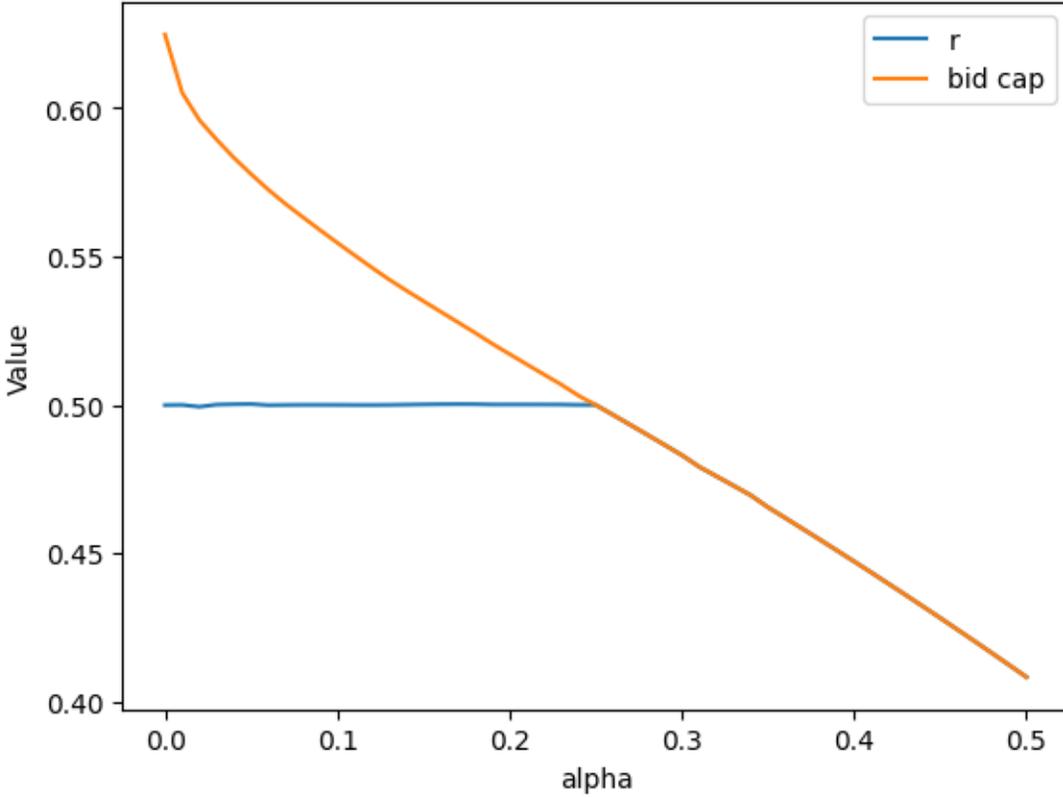


Figure 1: An example that illustrates the relationship between α and the object maximizing bid space. As long as the optimal bid cap is more than 0.5, the optimal reserve price is set at 0.5. Once α becomes sufficiently large, the optimal mechanism transitions to a first-price auction with only one bid amount is allowed, which is essentially a posted-price mechanism.

In the equilibrium, bidders of type 0.5 are indifferent between placing the the equilibrium bid and placing a bid of $\frac{3p+1}{4p+4}$.

4.3 Second Price Auctions

Can we use second price auctions in lieu of first price auctions in Theorem 1? Can we implement the outcomes of first-price auctions in second-price auctions? As the following example illustrates, this is in general not possible. In particular, the above proposition shows that with two types, an analogue of revenue equivalence obtains: same objectives can be achieved by first-price and second-price auctions. With three or more types, the revenue equivalence fails and the following example shows that revenue in a first-price auction might be strictly higher the maximum revenue achievable in second-price auctions.

Example 3. There are 2 bidders, and each bidder i 's value θ_i is drawn independently following

$$\theta_i = \begin{cases} 1 & \text{with probability 0.6,} \\ 11/4 & \text{with probability 0.2,} \\ 13/2 & \text{with probability 0.2.} \end{cases}$$

Suppose $\mathcal{B} = \{1, 2, 3\}$. Then,

$$\sigma(\theta_i) = \begin{cases} 1 & \text{if } \theta_i = 1, \\ 2 & \text{if } \theta_i = 11/4, \\ 3 & \text{if } \theta_i = 13/2. \end{cases}$$

is a symmetric equilibrium of the first-price auction with designed bids \mathcal{B} and

$$Q(\theta) = \begin{cases} 0.6 & \text{if } \theta = 1, \\ 0.8 & \text{if } \theta = 11/4, \\ 1 & \text{if } \theta = 13/2. \end{cases}$$

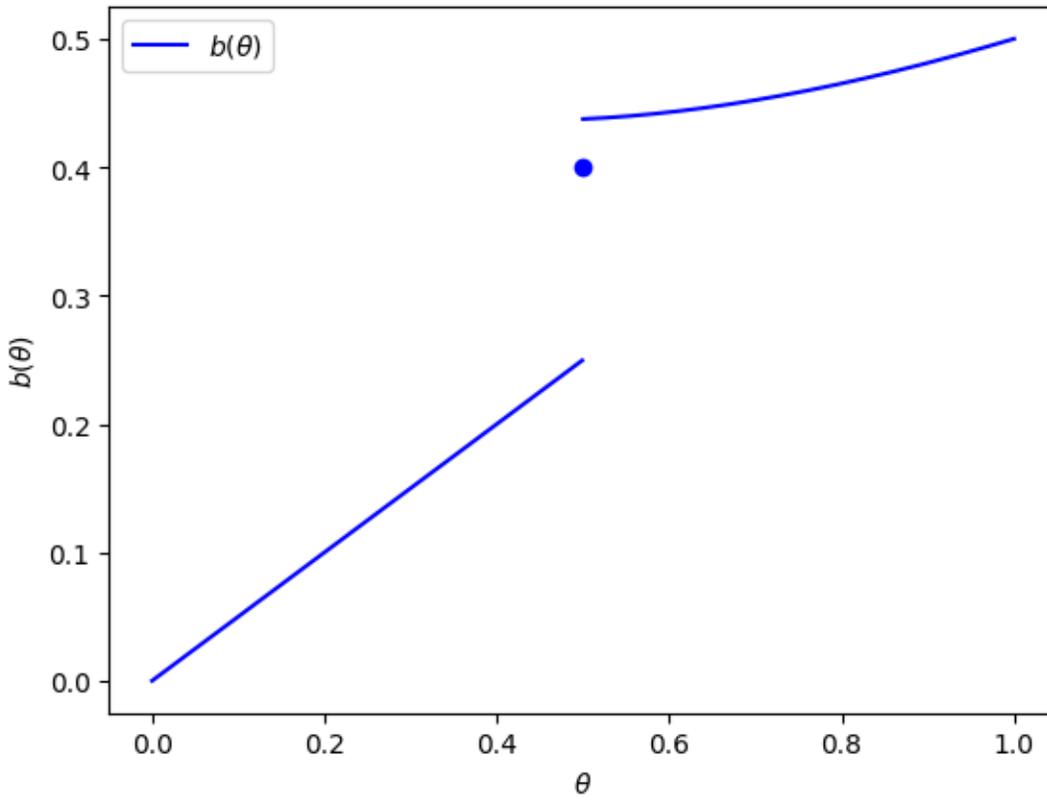
The expected payment for each type is

$$T(1) = 0.3, T(11/4) = 1.4, T(13/2) = 2.7.$$

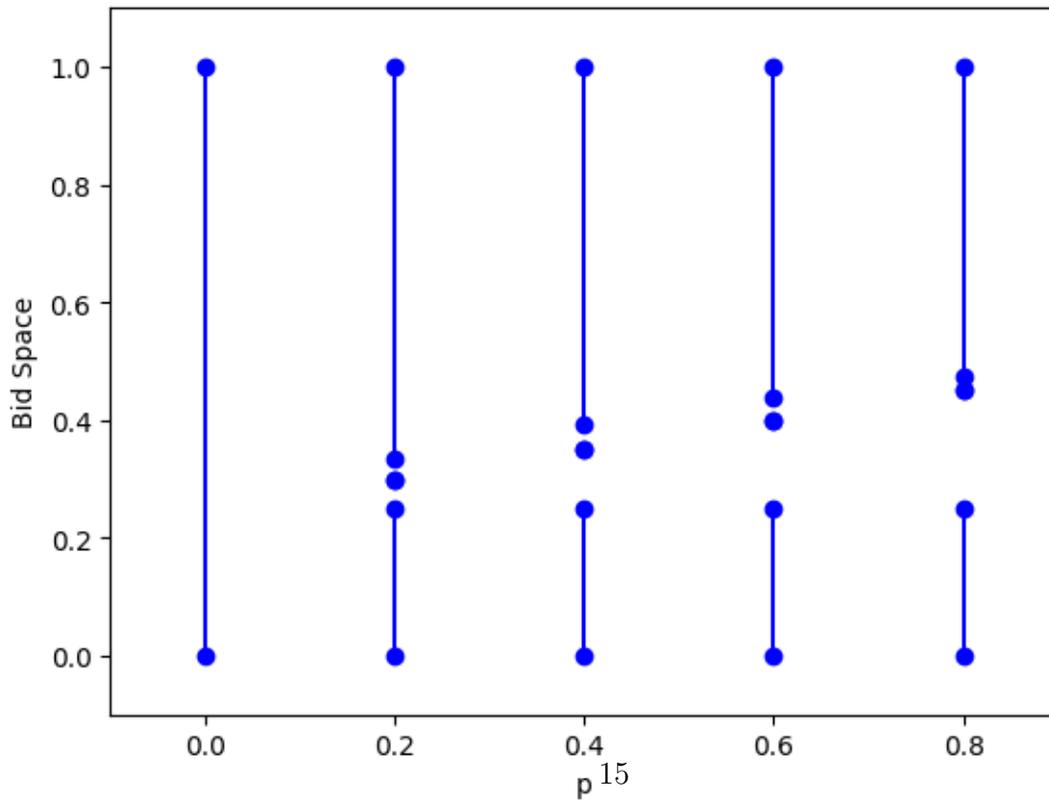
Now, consider a second-price auction to check if there exists a bids space such that the second-price auction achieves the same outcome with. Since the lowest type's utility in the

Figure 2: Example 2

(a) The equilibrium bidding strategy when $p = 0.6$



(b) The efficient bid space for varying values of probability p of an atom at .5.



equilibrium is 0, the equilibrium bid level for the lowest type is 1. The expected payment for the middle type when he places a bid level b_M is

$$b_L \times 0.6 + b_M \times \frac{1}{2} \times 0.2 = 0.6 + b_M \times \frac{1}{2} \times 0.2.$$

In order to match the expected payment under the second-price auction to the one in the first-price auction, b_M must solve the equation

$$0.6 + b_M \times \frac{1}{2} \times 0.2 = 1.4$$

and $b_M = 8$. We cannot separate the highest type from the middle type because a bidder with the value $13/2$ will never place a bid above 8 in a symmetric equilibrium.

The idea behind of the example is as follows. In a second price auction with designed bids, the designer can make a bidder to place a bid above their own value. If the bidder pays the second highest bid with very high probability, the level of bid the bidder is willing to place can be arbitrarily large. In the example, the bid level that a middle type is willing to place is even above the value of the high type and it makes the high type deviates from the bid he was suppose to submit.

Example 3 above relied on there being three types. With two types only, second-price auctions become sufficient to implement any objective. This implied by the following

Proposition 1. Any expected allocation and payment in a pure symmetric equilibrium in a first-price auction can be implemented by a second-price auctions if $|\Theta| \leq 2$.

Proof. Suppose that $\Theta = \{\theta_L, \theta_H\}$ and $\theta_H > \theta_L$. Denote the probability that a bidder has a type θ_H by π . Given a symmetric equilibrium in the first-price auction, denote the expected allocation and the expected payment for a bidder with type θ by $Q(\theta)$ and $T(\theta)$. Note that $Q(\theta_H) \geq Q(\theta_L)$. If $Q(\theta_L) = Q(\theta_H) = 0$, define $\mathcal{B} = \{b, \emptyset\}$, where b is any number strictly greater than θ_H . Then, the only equilibrium strategy in the second-price auction with \mathcal{B} is to choose \emptyset . If $Q(\theta_L) = 0$ and $Q(\theta_H) > 0$, define $\mathcal{B} = \{b, \emptyset\}$, where $b = \frac{T(\theta_H)}{Q(\theta_H)}$. Then, it is a symmetric equilibrium that the bidders with the higher type choose b and the bidders with the lower type choose \emptyset . If $Q(\theta_H) = Q(\theta_L) > 0$, define $\mathcal{B} = \{b, \emptyset\}$, where $b = \frac{T(\theta_H)}{Q(\theta_H)}$. Then, it is a symmetric equilibrium that all the bidders choose b . Suppose that $Q(\theta_H) > Q(\theta_L) > 0$. Define $\mathcal{B} = \{b_L, b_H, \emptyset\}$, where $b_L = \frac{T(\theta_L)}{Q(\theta_L)}$ and $b_H = \frac{T(\theta_H) - (n-1)T(\theta_L)}{Q(\theta_H) - (n-1)Q(\theta_L)}$ and consider a strategy where the bidders with the lower type chooses b_L and the bidders with the higher type choose b_H . We show that (i) b_H is well-defined, (ii) $b_H > b_L$, and (iii) it is an equilibrium that the bidders with the lower type choose b_L and the bidders with the higher type choose b_H .

(i) Note that $Q(\theta_H) = \sum_{m=0}^{n-1} \frac{1}{m+1} \binom{n-1}{m} \pi^m (1-\pi)^{n-m-1} = \sum_{m=1}^{n-1} \frac{1}{m+1} \binom{n-1}{m} \pi^m (1-\pi)^{n-m-1} + (1-\pi)^{n-1}$ and $Q(\theta_L) = \frac{1}{n-1} (1-\pi)^{n-1}$. Therefore, $Q(\theta_H) > (n-1)Q(\theta_L)$.

(ii) Note that $[Q(\theta_H) - (n-1)Q(\theta_L)] * b_L + (n-1)T(\theta_L) = Q(\theta_H)T(\theta_L)/Q(\theta_L) < T(\theta_H)$, which implies $b_L < b_H$.

(iii) It is trivial that the expected payment of a low-type bidder in the given strategy in the second-price auction is $T(\theta_L)$. For a bidder with the type θ_H , he pays b_L only when all the other bidders have the lower type. Therefore, the expected payment of a bidder with the higher type is $(1-\pi)^{n-1}b_L + (Q(\theta_H) - (1-\pi)^{n-1})b_H = T(\theta_H)$. Because the expected winning probability and the expected payment for each type in the suggested strategy in the second-price auction coincide with the ones in the first-price auction, the suggested strategy is indeed an equilibrium strategy. \square

Finally, let us observe that the expected allocations and payments in any second price auction can be replicated in a first price auction.

Proposition 2. Any expected allocation and payment in a symmetric equilibrium in a second-price auction can be implemented by a first-price auction.

Proof. Given a symmetric equilibrium strategy σ_{SPA} in the second-price auction with the bid space \mathcal{B}_{SPA} , we use $Q^{SPA}(b)$ and $T^{SPA}(b)$ to denote the expected winning probability and the expected payment of a bidder when he plays a bid $b \in \mathcal{B}_{SPA}$ assuming that all the other bidders follow the strategy σ_{SPA} . Without loss of generality, let us assume that for any $b \in \mathcal{B}_{SPA}$, there exists $\theta \in \Theta$ such that $b \in \sigma_{SPA}(\theta)$.¹⁵ Define a mapping $X : \mathcal{B}_{SPA} \rightarrow \mathbb{R} \cup \{\emptyset\}$ by

$$X(b) = \begin{cases} \frac{T^{SPA}(b)}{Q^{SPA}(b)} & \text{if } b \neq \emptyset \\ \emptyset & \text{if } b = \emptyset \end{cases}.$$

We construct the bid space for a first-price auction and a strategy; $\mathcal{B}_{FPA} = \{X(b) | b \in \mathcal{B}_{SPA}\} \cup \{\emptyset\}$ and $\sigma_{FPA}(\theta) = X(\sigma_{SPA}(\theta))$.¹⁶ Denote the expected winning probability and the expected payment of a bidder playing $b \in \mathcal{B}_{FPA}$ by $Q^{FPA}(b)$ and $T^{FPA}(b)$ assuming that the other bidders follow the strategy σ_{FPA} . To show that σ_{FPA} is a symmetric equilibrium strategy in the first-price auction, we only need to check that the mapping $X(b)$ is strictly increasing in b for all $b \in \mathcal{B}_{SPA} - \{\emptyset\}$ as then $Q^{FPA}(X(b)) = Q^{SPA}(b)$ and $T^{FPA}(X(b)) = T^{SPA}(b)$ for all $b \in \mathcal{B}_{SPA}$, thus the incentive properties of the second price auction are inherited by the first price auction we construct.¹⁷

¹⁵Otherwise, we can use a subset of \mathcal{B}_{SPA} to construct a bid space and an symmetric equilibrium in the first-price auction.

¹⁶With a slight abuse of notation, $X(\sigma_{SPA}(\theta))$ denotes the mixed strategy, where the probability of playing a bid $b \in \sigma_{SPA}(\theta)$ is equal to the the probability of playing $X(b)$ in the mixed strategy σ_{FPA} .

¹⁷The weak monotonicity would not be sufficient for this claim because of the possible tie-breaking.

For the monotonicity verification, take any two arbitrary b and b' in $\mathcal{B}_{SPA} - \{\emptyset\}$ such that $b' > b$. Then

$$Q^{SPA}(b)\sigma_{SPA}^{-1}(b) - T^{SPA}(b) \geq Q^{SPA}(b')\sigma_{SPA}^{-1}(b) - T^{SPA}(b'),$$

which is equivalent to

$$\frac{Q^{SPA}(b)}{Q^{SPA}(b')} [\sigma_{SPA}^{-1}(b) - \frac{T^{SPA}(b)}{Q^{SPA}(b)}] \geq \sigma_{SPA}^{-1}(b) - \frac{T^{SPA}(b')}{Q^{SPA}(b')}.$$

Because $0 < Q^{SPA}(b) < Q^{SPA}(b')$, we can conclude that

$$\frac{T^{SPA}(b')}{Q^{SPA}(b')} > \frac{T^{SPA}(b)}{Q^{SPA}(b)},$$

thus $X(\cdot)$ is strictly increasing as required. □

5 Application: Credible Implementation

We now study that how the auctioneer can use a first-price auction to achieve his goal credibly in the sense of Akbarpour and Li (2020). Mapping their terminology to our study of auctions, we say that an observation of bidder i is (a_i, p_i) , where, as above, p_i is the payment of bidder i and $a_i = 1$ iff bidder i wins, and it equals 0 otherwise. Given an allocation rule (\mathbf{a}, \mathbf{p}) and equilibrium σ , another allocation rule $(\hat{\mathbf{a}}, \hat{\mathbf{p}})$ is safe if for every bidder i and every profile of types $(\theta_1, \dots, \theta_n)$, the observation of i given the allocation $(\hat{\mathbf{a}}, \hat{\mathbf{p}}) \sigma(\theta_1, \dots, \theta_n)$ is the same as the observation of this bidder given allocation $(\mathbf{a}, \mathbf{p}) \sigma(\theta_i, \theta'_{-i})$ for some profile of types θ'_{-i} of bidders other than i . A first-price auction is credible if no safe allocation rule may strictly improve the objective at some profile of types. Akbarpour and Li (2020) showed that first-price auctions are credible when the objective is a weighted average of revenue and welfare; using their approach, one can easily show a slightly more general result.

Lemma 3. Suppose that $\Theta \subset \mathbb{R}_+$ and the bidders play a symmetric equilibrium. If a standard objective is a function of revenue and social welfare, and it is weakly increasing in each argument, then any first-price auctions with \mathcal{B} such that $\mathcal{B} \cap \mathbb{R} \subseteq \mathbb{R}_+$ is credible.

The restriction on the bid space $\mathcal{B} \cap \mathbb{R} \subseteq \mathbb{R}_+$ matters as in its absence there might be a safe deviation in which the item is not allocated when all bids are negative.

Proof. Denote the winner by i in a given first-price auction by following the rule. Suppose

that the auctioneer safely deviate by picking a winner j , which is not i . If $b_j = b_i$, then it is easy to see that the revenue does not change. Moreover, the social welfare does not change either since the auctioneer cannot distinguish the types among the bidders who bid the same. If $b_j < b_i$, then the revenue decreases and the social welfare does not increase since any symmetric equilibrium is weakly monotonic. The safe deviation that does not sell the item does not increase the objective because we assume that the types and real bids are positive. \square

This lemma and our 1 imply the following.

Corollary 2. If a standard objective is a function of revenue and social welfare, and it is increasing in each argument, then this objective can be credibly achieved by a first-price auction.

Proof. By 1, a first-price auction maximizes the objective. We can restrict our attention to the environment where $\Theta \in \mathbb{R}_+$ and only positive bids are allowed without loss of generality, because if a bidder with a negative type wins the item and pays a negative bid, the auctioneer can increase the objective by setting $\min \mathcal{B} \cap \mathbb{R}$ to be non-negative. Then, the corollary is a direct implication of 3. \square

While one may conjecture that the above results can extend to any standard objectives, the following example shows that some standard objectives cannot be credibly achieved by a first-price auction.

Example 4. There are two bidders and each bidder's type is drawn from $\Theta = \{0, 1\}$ uniformly at random. While the auctioneer prefers more revenue, he also has a distributional concern. Specifically, the objective of the auctioneer is $\mathbb{E}_\theta[t_1(\theta) + t_2(\theta)]$ if a bidder whose type is 1 wins the item, and $\mathbb{E}_\theta[t_1(\theta) + t_2(\theta)] + A$ if a bidder whose type is 0 wins the item. One can verify that, with $A \in [1, 2]$, the first-price auction with $\mathcal{B} = \{0, \frac{2}{3}\}$ maximizes the objective with the symmetric equilibrium $\sigma(0) = 0$ and $\sigma(1) = \frac{2}{3}$ and the resulting utility of the auctioneer is $\frac{1}{2} + \frac{1}{4}A$. Suppose that auctioneer deviates by allocating the item to a bidder that bid 0 when the other bidder bid 1. Note that this deviation is safe and the resulting utility is $\frac{1}{6} + \frac{3}{4}A$, which is strictly greater than $\frac{1}{2} + \frac{1}{4}A$ for $A > \frac{2}{3}$.

6 Extreme Points of Reduced Form Direct Mechanisms with Finite Type Spaces

In this auxiliary section, we restrict attention to finite type spaces and study the set of incentive compatible symmetric social rules and show that each extreme point of the set

takes a special form that can be generated by a first-price auction. A natural class of symmetric social rule allocates the object with equal probabilities to the bidders with the highest type. We generalize this class by allowing equal-probability allocation to bidders in a highest interval of types (which call the highest tier) as follows. We say that a partition $\Theta_1, \Theta_2, \dots, \Theta_J$ of Θ is monotonic partition if $\max \Theta_j < \min \Theta_{j+1}$ for all $j = 1, \dots, J - 1$ and we say that a social rule \mathbf{y} is monotonic hierarchical if there exists a monotonic partition $\Theta_1, \Theta_2, \dots, \Theta_J$ of Θ such that for all i , one of the following two possibilities obtains:

$$q_i(\theta_1, \dots, \theta_n) = \begin{cases} \frac{1}{|\{l: \theta_l \in \Theta_j\}|}, & \text{if } \theta_i \in \Theta_j \text{ and } \theta_{i'} \notin \cup_{l=j+1}^J \Theta_l \text{ for all } i' \in N, \\ 0, & \text{otherwise,} \end{cases}$$

or

$$q_i(\theta_1, \dots, \theta_n) = \begin{cases} \frac{1}{|\{n: \theta_n \in \Theta_j\}|}, & \text{if } j \geq 2, \theta_i \in \Theta_j \text{ and } \theta_{i'} \notin \cup_{l=j+1}^J \Theta_l \text{ for all } i' \in N, \\ 0, & \text{otherwise.} \end{cases}$$

The former possibility encompasses social rules that always allocate the object, while the latter possibility encompasses social rules in which the object is unallocated if no bidder's type is above a certain threshold. The popular implementation of the latter of social rules is via a reserve price. More generally, in the appendix, we prove the following:

Theorem 5. Any implementable, individually rational, and monotonic hierarchical social rule is generated by a first-price auction.

This theorem, together with the decomposition result we state next, allows us to prove the key results of Section 3.

Theorem 6. Any implementable and individually rational social rule is a convex combination of implementable, individually rational, and monotonic hierarchical social rules.

In particular, this theorem implies that if a social rule \mathbf{y} is an extreme point of the space of symmetric, implementable, and individually rational social rules then \mathbf{y} is monotonic hierarchical. Theorem 6 extends an insight of Manelli and Vincent (2010), who studied continuous type spaces. We establish this theorem as immediate consequence of the following two lemmas, proven in the appendix. In proving the first lemma—but not the second—we are able to leverage the approach that Manelli and Vincent (2010) employed in their analysis.¹⁸

¹⁸Earlier developments of Manelli and Vincent's structure theorem, most notably Gershkov et al. (2013), analyzed discrete type spaces, but their approach was different from ours; in particular, they did not characterize incentive-compatible mechanisms in terms of convex combinations of simpler classes of rules, and neither Lemma 4 nor Lemma 5 has a counterpart in their analysis.

In order to prove the theorem 6, we study the space, denoted by \mathcal{Q} of non-decreasing mappings from Θ to $[0, 1]$, where each mapping also satisfies the condition in the lemma 9. The first lemma shows that any extreme point of the space is an interim expected allocation of a monotonic hierarchical social rule.

Lemma 4. The following three conditions are equivalent:

- (i) Q is an extreme point of \mathcal{Q} .
- (ii) There is a monotonic hierarchical rule \mathbf{y} such that $(Q, \dots, Q) \in \mathcal{Q}^n$ is an interim allocation rule of \mathbf{y} .
- (iii) The coarsest monotonic partition $\{\Theta_1, \dots, \Theta_J\}$ of Θ that is adapted to Q satisfies

$$Q(\Theta_j) = \frac{[\sum_{l=1}^j \pi(\Theta_l)]^n - [\sum_{l=1}^{j-1} \pi(\Theta_l)]^n}{n\pi(\Theta_j)}, \text{ for } j = 2, \dots, J, \quad (1)$$

$$Q(\Theta_1) = 0 \text{ or } \frac{\pi(\Theta_1)^{n-1}}{n}. \quad (2)$$

This characterization of extreme points implies that, in our environment, any symmetric interim allocation rule is a convex combination of interim allocation rules of hierarchical social rules. To extend this insight to mechanisms, that is allocation rules and payment rules, we use Farkas Alternative to prove the following decomposition lemma. This lemma has no direct counterpart in Manelli and Vincent (2010), who focus on allocations.¹⁹

Lemma 5. (Decomposition Lemma) Consider a reduced-form $(\mathbf{Q}, \mathbf{T}) = ((Q_1, T_1), \dots, (Q_n, T_n))$ of a social rule that is incentive compatible, individually rational. Suppose that $Q_i = \sum_{k=1}^K \alpha_k Q_i^k$ for some Q_i^k s and weights $\alpha_k > 0$ such that $\sum_{k=1}^K \alpha_k = 1$, for all $i \in N$. Then, $Q_i^1(\theta), \dots, Q_i^K(\theta)$ are non-decreasing in θ if and only if there exist T_i^1, \dots, T_i^K such that (Q_i^k, T_i^k) is incentive compatible, individually rational, and $T_i = \sum_{k=1}^K \alpha_k T_i^k$.

References

- Akbarpour, Mohammad, Piotr Dworzak, and Scott Duke Kominers (2020). Redistributive Allocation Mechanisms. Working Paper.
- Akbarpour, Mohammad and Shengwu Li (2020). “Credible Auctions: A Trilemma”. In: *Econometrica* 88.2, pp. 425–467.

¹⁹As they observe, in the continuous environment they study, the payment functions can be reconstructed, up to a constant, from incentive compatibility conditions. In our discrete environment such a recovery is not possible.

- Bergemann, Dirk and Martin Pesendorfer (2007). “Information structures in optimal auctions”. In: *Journal of Economic Theory* 137.1, pp. 580–609.
- Bergson, Abram (1938). “A Reformulation of Certain Aspects of Welfare Economics”. In: *The Quarterly Journal of Economics* 52.2, pp. 310–334.
- Bogachev, V.I. (2006). *Measure Theory*. Springer Berlin Heidelberg.
- Border, Kim C. (1991). “Implementation of Reduced Form Auctions: A Geometric Approach”. In: *Econometrica* 59.4, pp. 1175–1187.
- (2015). *Alternative Linear Inequalities*.
- Carbajal, Juan Carlos and Jeffrey C. Ely (Jan. 2013). “Mechanism design without revenue equivalence”. In: *Journal of Economic Theory* 148.1, pp. 104–133.
- Carroll, Gabriel (2012). “When Are Local Incentive Constraints Sufficient?” In: *Econometrica* 80.2, pp. 661–686.
- Che, Yeon-Koo and Ian Gale (1998a). “Caps on Political Lobbying”. In: *The American Economic Review* 88.3, pp. 643–651.
- (1998b). “Standard Auctions with Financially Constrained Bidders”. In: *The Review of Economic Studies* 65.1, pp. 1–21.
- (2006). “Revenue comparisons for auctions when bidders have arbitrary types”. In: *Theoretical Economics* 1.1, pp. 95–118.
- Chung, Kim-Sau and Wojciech Olszewski (2007). “A non-differentiable approach to revenue equivalence”. In: *Theoretical Economics* 2.4, pp. 469–487.
- Condorelli, Daniele (2013). “Market and non-market mechanisms for the optimal allocation of scarce resources”. In: *Games and Economic Behavior* 82, pp. 582–591.
- Gershkov, Alex et al. (2013). “On the Equivalence of Bayesian and Dominant Strategy Implementation”. In: *Econometrica* 81.1, pp. 197–220.
- Heydenreich, Birgit et al. (2009). “Characterization of Revenue Equivalence”. In: *Econometrica* 77.1, pp. 307–316.
- Kleiner, Andreas, Benny Moldovanu, and Philipp Strack (2021). “Extreme Points and Majorization: Economic Applications”. In: *Econometrica* 89.4, pp. 1557–1593.
- Lovejoy, William S. (2006). “Optimal Mechanisms with Finite Agent Types”. In: *Management Science* 52.5, pp. 788–803.
- Manelli, Alejandro M. and Daniel R. Vincent (2010). “Bayesian and Dominant-Strategy Implementation in the Independent Private-Values Model”. In: *Econometrica* 78.6, pp. 1905–1938.
- Maskin, Eric and John Riley (1984). “Optimal Auctions with Risk Averse Buyers”. In: *Econometrica* 52.6, pp. 1473–1518.

- Matthews, Steven A. (1984). “On the Implementability of Reduced Form Auctions”. In: *Econometrica* 52.6, pp. 1519–1522.
- May, Kenneth O. (1952). “A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision”. In: *Econometrica* 20.4, pp. 680–684.
- McAfee, R.Preston and John McMillan (1988). “Multidimensional incentive compatibility and mechanism design”. In: *Journal of Economic Theory* 46.2, pp. 335–354.
- Mirrlees, J.A. (1976). “Optimal tax theory: A synthesis”. In: *Journal of Public Economics* 6.4, pp. 327–358.
- Mishra, Debasis, Anup Pramanik, and Souvik Roy (2016). “Local incentive compatibility with transfers”. In: *Games and Economic Behavior* 100, pp. 149–165.
- Myerson, Roger B. (1981). “Optimal Auction Design”. In: *Mathematics of Operations Research* 6.1, pp. 58–73.
- Nikzad, Afshin (July 2022). “Constrained Majorization: Applications in Mechanism Design”. In: *Proceedings of the 23rd ACM Conference on Economics and Computation*. ACM.
- Olszewski, Wojciech, Philip J Reny, and Ron Siegel (2023). Equilibrium existence in independent private-value first-price auctions. In Progress.
- Olszewski, Wojciech and Ron Siegel (2019). “Bid caps in large contests”. In: *Games and Economic Behavior* 115, pp. 101–112.
- Pai, Mallesh M. and Rakesh Vohra (2014). “Optimal auctions with financially constrained buyers”. In: *Journal of Economic Theory* 150, pp. 383–425.
- Pycia, Marek (2019). Evaluating with Statistics: Which Outcome Measures Differentiate Among Matching Mechanisms? Working Paper.
- Reny, Philip J. (1999). “On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games”. In: *Econometrica* 67.5, pp. 1029–1056.
- (2011). “On the Existence of Monotone Pure-Strategy Equilibria in Bayesian Games”. In: *Econometrica* 79.2, pp. 499–553.
- Reny, Philip J. and Shmuel Zamir (2004). “On the Existence of Pure Strategy Monotone Equilibria in Asymmetric First-Price Auctions”. In: *Econometrica* 72.4, pp. 1105–1125.
- Riley, John G. and William F. Samuelson (1981). “Optimal Auctions”. In: *The American Economic Review* 71.3, pp. 381–392.
- Rochet, Jean-Charles (1987). “A necessary and sufficient condition for rationalizability in a quasi-linear context”. In: *Journal of Mathematical Economics* 16.2, pp. 191–200.
- Rockafellar, R. Tyrrell and Roger J.-B. Wets (1998). *Variational Analysis*. Heidelberg, Berlin, New York: Springer Verlag.
- Samuelson, Paul A. (1956). “Social Indifference Curves”. In: *The Quarterly Journal of Economics* 70.1, pp. 1–22.

- Sen, Amartya (1970). *Collective Choice and Social Welfare*. San Francisco: Holden Day.
- Spence, Michael (1974). “Competitive and optimal responses to signals: An analysis of efficiency and distribution”. In: *Journal of Economic Theory* 7.3, pp. 296–332.
- Szech, Nora (2015). “Tie-breaks and bid-caps in all-pay auctions”. In: *Games and Economic Behavior* 92, pp. 138–149.
- Vohra, Rakesh V. (2005). *Advanced Mathematical Economics*. Advanced Texts in Economics Series. Routledge.
- Woodward, Kyle (2016). “Strategic Ironing in Pay-as-Bid Auctions: Equilibrium Existence with Private Information”. In: *Proceedings of the 2016 ACM Conference on Economics and Computation*. EC '16. Maastricht, The Netherlands: Association for Computing Machinery, p. 121.

A Proofs

We rely on two well-known results from linear programming.

Lemma 6. (Farkas’ Alternative)²⁰ Suppose that $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{l \times n}$, and $d \in \mathbb{R}^l$. Then exactly one of the following two statements is true:

- There exists $x \in \mathbb{R}^n$ such that $Cx = d$ and $Ax \leq b$.
- There exist $y \in \mathbb{R}^l$ and $z \in \mathbb{R}^m$ such that $zA + yC = 0$, $z \geq 0$, and $z \cdot b + y \cdot d < 0$.

Lemma 7. (Fundamental Theorem of Linear Programming)²¹ Suppose that $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Consider a set $P = \{x \in \mathbb{R}^n | Ax \leq b\}$. Then, $x^* \in P$ is an extreme point of P if and only if $Ax^* \leq b$ has n binding constraints that are linearly independent.

We also rely on the following well-known results in mechanism design.

Lemma 8. (Characterizations of Incentive Compatibility)

- (Spence-Mirrlees Condition)²² For every interim probability rule \mathbf{Q} , there exists an interim payment rule \mathbf{T} such that the reduced-form mechanism (\mathbf{Q}, \mathbf{T}) is incentive compatible if and only if \mathbf{Q} is non-decreasing.

²⁰See Border (2015) for the this version of Farkas’ Alternative.

²¹See R. V. Vohra (2005).

²²For more detailed discussions, see Spence (1974), Mirrlees (1976), and Rochet (1987).

- (Local IC guarantees Global IC)²³ The reduced-form mechanism (\mathbf{Q}, \mathbf{T}) is incentive compatible and individually rational if and only if for all $i \in N$,

$$Q_i(\theta^t)\theta^t - T_i(\theta^t) \geq Q_i(\theta^{t-1})\theta^t - T_i(\theta^{t-1}), \text{ for all } t = 1, \dots, T, \quad (3)$$

$$Q_i(\theta^{t-1})\theta^{t-1} - T_i(\theta^{t-1}) \geq Q_i(\theta^t)\theta^{t-1} - T(\theta^t), \text{ for all } t = 2, \dots, T. \quad (4)$$

Because we assumed that $Q_i(\theta^0) = 0$ and $T_i(\theta^0) = 0$ (cf. footnote 4), the condition (3) encompasses individual rationality for type θ^1 .

Lemma 9. (Maskin-Riley-Matthews-Border Condition)²⁴ A symmetric interim allocation rule $\mathbf{Q} = (Q, \dots, Q)$ is feasible if and only if

$$n \sum_{\{\theta \in \Theta: Q(\theta) \geq \alpha\}} Q(\theta)\pi(\theta) \leq 1 - \left(\sum_{\{\theta \in \Theta: Q(\theta) < \alpha\}} \pi(\theta) \right)^\alpha, \text{ for all } \alpha \in [0, 1]. \quad (5)$$

Then, the above lemmas tell us that

$$\mathcal{Q}^S = \{(Q, \dots, Q) : \Theta^n \rightarrow [0, 1]^n \mid Q \text{ satisfies (5) and non-decreasing}\}. \quad (6)$$

Because the type space is finite, for any $(Q, \dots, Q) \in \mathcal{Q}^S$ there is a partition $\{\Theta_1, \dots, \Theta_J\}$ of Θ such that $Q(\theta) = Q(\theta')$ for all $\theta, \theta' \in \Theta_j$. We then say that the partition is adapted to Q and write $Q(\Theta_j)$ to denote the common value $Q(\theta)$ of all $\theta \in \Theta_j$.

Proof of Theorem 1

For finite type spaces, it is a direct implication of Theorem 3. Let us consider the general case of Θ being a compact subset of \mathbb{R} . The linear combinations of Dirac measures are dense in the measure space on Θ equipped with Weak-*Topology.²⁵ Thus, there exists a sequence $(\mu_k)_{k \in \mathbb{N}}$ of probability measures with finite supports that converges to μ ; we denote by Θ_k the support of μ_k . Because the set of atoms of μ is at most countable, we can ensure that for any atom θ in μ there is $k(\theta)$ such that for all $k \geq k(\theta)$ we have $\mu_k(\theta) = \mu(\theta)$; in particular, μ_k then also has an atom at θ . By the discrete version of Theorem 1, for each μ_k ,

²³For more detailed discussions, see McAfee and McMillan (1988), Carroll (2012), and Mishra, Pramanik, and Roy (2016). The equivalence of local and global incentive compatibility conditions relies on the single crossing property, which is satisfied in our setting.

²⁴Maskin and J. Riley (1984) studied the condition when the type space is continuous and the allocation is a step function in their analysis of optimal auctions with risk-averse bidders. Matthews (1984) showed that the condition holds for any increasing allocation functions. Border (1991) proved the condition for general type spaces and allocation functions.

²⁵See Example 8.1.6 (i) in Bogachev (2006) for the proof.

there is FPA with bid space B_k and symmetric equilibrium bid function $\tilde{b}_k : \Theta_k \rightarrow B_k$ that implements a social rule which maximizes the objective under μ_k . By incentive compatibility of the equilibrium bid, \tilde{b}_k is weakly increasing. We extend the domain of \tilde{b}_k to Θ by defining

$$b_k(\theta) \in \arg \max_{b \in B_k} P_k(b)(\theta - b),$$

where $P_k(b) = \sum_{n=1}^N \frac{1}{n} \binom{N-1}{n-1} \left(\mu_k(\{\theta | b > \tilde{b}_k(\theta)\}) \right)^{N-n} \left(\mu_k(\{\theta | b = \tilde{b}_k(\theta)\}) \right)^{n-1}$ is the probability of winning against Θ_k bidders bidding \tilde{b}_k . By incentive compatibility of \tilde{b}_k , for all $\theta \in \Theta_k$ we can select $b_k(\theta)$ so that $b_k(\theta) = \tilde{b}_k(\theta)$. We can further select b_k that is weakly increasing because if a type θ bidder weakly prefers b to b' , where $b > b'$, then any type above θ weakly prefers b to b' .

By construction $b_k(\cdot)$ is weakly increasing and uniformly bounded, thus, we can apply Helly's selection theorem to conclude that the sequence of functions $(b_k)_{k \in \mathbb{N}}$ admits a point-wise convergent subsequence. Let b be the limit of this subsequence; note that b is also weakly increasing. Without loss of generality, in the sequel we assume that $(b_k)_{k \in \mathbb{N}}$ itself is this convergent subsequence.

In order to show that $b(\cdot)$ is a bidding strategy in a symmetric equilibrium in the limit, notice that it is enough to show that $P_k(b_k(\theta)) \rightarrow P(b(\theta))$ for all θ , where $P(b(\theta))$ denotes the winning probability by bidding $b(\theta)$ when all the other bidders follow the same bidding strategy $b(\cdot)$ under the probability measure μ . We show that $\mu_k(\{x | b_k(x) < b_k(\theta)\})$ and $\mu_k(\{x | b_k(x) = b_k(\theta)\})$ converge to $\mu(\{x | b(x) < b(\theta)\})$ and $\mu(\{x | b(x) = b(\theta)\})$, which is followed by the convergence of $P_k(b_k(\theta))$ to $P(b(\theta))$.

Claim 1. We have $\mu_k(I) \rightarrow \mu(I)$ for subset $I \subseteq \Theta$ if the boundary ∂I is countable.

If I itself is countable, then the convergence is directly implied by the construction of μ_k . Suppose that I is uncountable and ∂I is countable. Notice that $\liminf \mu_k(I) \geq \liminf \mu_k(I - (I \cap \partial I)) + \liminf \mu_k(I \cap \partial I)$. By the Portmanteau theorem, $\liminf \mu_k(I - (I \cap \partial I)) \geq \mu(I - (I \cap \partial I))$. Because $I \cap \partial I$ is countable, $\liminf \mu_k(I \cap \partial I) = \mu(I \cap \partial I)$. Combining these establishes $\liminf \mu_k(I) \geq \mu(I)$.

Similarly, $\limsup \mu_k(I) \leq \limsup \mu_k(I \cup (\partial I - I)) + \limsup [-\mu_k((\partial I - I))]$. By applying the Portmanteau theorem and using the convergence of $\mu_k((\partial I - I))$, it follows that $\limsup \mu_k(I) \leq \mu(I)$. Therefore, $\lim_k \mu_k(I) = \mu(I)$ if ∂I is countable.

Claim 2. For any $\theta \in \Theta$, $\limsup_k I_k(\theta) \subseteq I(\theta)$.

Proof of Claim 2: Suppose that $x \in \limsup_k I_k(\theta)$. Because $b_k(x) = b_k(\theta)$ infinitely often and $b_k(\cdot)$ converges to $b(\cdot)$, $b(x) = b(\theta)$, thus, $x \in I(\theta)$.

Claim 3. Suppose that $\mu(I(\theta)) > 0$ and for all $x \in I(\theta)$, $b(x) < x$. Then, $I(\theta) \subseteq \liminf_k I_k(\theta)$.

Proof of Claim 3: Suppose that there exists $\theta' \in I(\theta) - \liminf_k I_k(\theta)$. Notice that $\theta' \neq \theta$ because $\liminf_k I_k(\theta)$ includes θ . For any K , there exists $k > K$ such that $b_k(\theta') \neq b_k(\theta)$. The monotonicity of b_k allows us to pick $x, y \in I(\theta)$ such that for any K , there exists $k > K$ such that $b_k(x) < b_k(y)$ and $\mu([x, y]) > 0$.

From the incentive compatibility for x , the following holds;

$$[P_k(b_k(y)) - P_k(b_k(x))] (x - b_k(x)) \leq P_k(b_k(y)) (b_k(y) - b_k(x)).$$

By the monotonicity of $b_k(\cdot)$, $b_k(y) \geq b_k(x)$ and $P_k(b_k(y)) \geq P_k(b_k(x))$. Because $b_k(y) - b_k(x)$ would get arbitrarily close to 0, so would $P_k(b_k(y)) - P_k(b_k(x))$. Suppose that we can find $k > K$ such that $b_k(y) - b_k(x) > 0$ for any K . The tie-breaking ensures that $[P_k(b_k(y)) - P_k(b_k(x))] \geq \frac{1}{N} \mu_k([x, y])^{N-1}$, thus, the limit of $[P_k(b_k(y)) - P_k(b_k(x))] (x - b_k(x))$ is bounded below by $\frac{1}{N} \mu([x, y])^{N-1} (x - b(x))$, and this leads to a contradiction.

Claim 4. $\mu_k(I_k(\theta)) \rightarrow \mu(I(\theta))$.

Proof of Claim 4: For any $\theta \in \Theta$ such that $\theta > b(\theta)$ and $\mu(I(\theta)) > 0$, notice that

$$\limsup_k I_k(\theta) \subseteq I(\theta) \subseteq \liminf_k I_k(\theta)$$

by Claim 2 and 3. Thus, the limit of $I_k(\theta)$ exists and equals $I(\theta)$. Then,

$$\mu(I(\theta)) = \lim_k \mu(\cup_{j \geq k} I_j(\theta)) = \lim_k \mu(\cap_{j \geq k} I_j(\theta)) = \lim_k \mu(I_k(\theta)).$$

Moreover, for any $\varepsilon > 0$, there exists K' such that $\mu(\cup_{j \geq K'} I_j(\theta) - I(\theta)) < \varepsilon$. Notice that, for any $k \geq K'$, $\mu_k(\cup_{j \geq k} I_j(\theta) - I(\theta)) \leq \mu_k(\cup_{j \geq K'} I_j(\theta) - I(\theta))$. As each $I_j(\theta)$ and $I(\theta)$ is an interval, Claim 1 implies that $\mu_k(\cup_{j \geq K'} I_j(\theta) - I(\theta))$ converges to $\mu(\cup_{j \geq K'} I_j(\theta) - I(\theta))$. Since $\lim_k \mu_k(\cup_{j \geq k} I_j(\theta) - I(\theta)) < \varepsilon$ and ε can be arbitrarily chosen, $\lim_{k \rightarrow \infty} \mu_k(\cup_{j \geq k} I_j(\theta) - I(\theta)) = 0$. Similarly, one can show that $\lim_k \mu_k(\cap_{j \geq k} I_j(\theta) - I(\theta)) = 0$, thus, $\lim_k \mu_k(\cap_{j \geq k} I_j(\theta)) = \lim_k \mu_k(\cup_{j \geq k} I_j(\theta)) = \lim_k \mu_k(I(\theta)) = \mu(I(\theta))$. Because $\mu_k(\cap_{j \geq k} I_j(\theta)) \leq \mu_k(I_k(\theta)) \leq \mu_k(\cup_{j \geq k} I_j(\theta))$, we obtain the claim $\lim_k \mu_k(I_k(\theta)) = \mu(I(\theta))$ under the assumptions above.

For $k \geq K'$ and $I(\theta)$ such that $\mu(I(\theta)) = 0$ (whether $\theta > b(\theta)$ or not), Claim 2 implies that $\limsup_k I_k(\theta) \subseteq I(\theta)$ and, thus, $\lim_k \mu_k(\cup_{j \geq k} I_j(\theta)) = 0$ and $\lim_k \mu_k(I_k(\theta)) \leq \lim_k \mu_k(\cup_{j \geq k} I_j(\theta)) = 0$. Because μ_k is a probability measure, $\lim_k \mu_k(I_k(\theta)) = 0$.

It remains to analyze the case $\mu(I(\theta)) > 0$ and there exists a type θ' in $I(\theta)$ such that $b(\theta') = \theta'$. The latter condition can only hold in equilibrium if θ' is the lowest type that has positive chance of winning in the limit. Notice that the similar proof of Claim 3 can be applied to the set $I(\theta) - \{\theta'\}$ so that $I(\theta) - \{\theta'\} \subseteq \liminf_k (I_k(\theta) - \{\theta'\})$. Therefore, if $\theta' \in \liminf_k I_k(\theta)$, thus, $I(\theta) \subseteq \liminf_k (I_k(\theta))$, then Claim 4 holds. Suppose that $\theta' \notin$

$\liminf_k I_k(\theta)$. If there is no μ -atom at θ' then the convergence of $b_k(\theta')$ to $b(\theta')$ does not affect the winning probability of other types in $I(\theta)$ in the limit and for bidders with type θ' , it is optimal to bid $b(\theta')$ as θ' is the lowest possible bid. Suppose that $\mu(\{\theta'\}) > 0$.

Because $\theta' \notin \liminf_k I_k(\theta)$, for any K , there exists $k > K$ such that $b_k(\theta') < b_k(\theta)$. For a sufficiently large K , $b_k(\theta)$ is arbitrarily close to θ' . If θ' is an isolated point in $I(\theta)$, thus all the types above θ' in $I(\theta)$ are bounded away from θ' , the designer can increase the revenue without altering the allocation by increasing all bid levels by $\varepsilon/P_k(b_k(\theta))$ for all $\theta \neq \theta'$, for some small ε , and this leads to a contradiction. Suppose that θ' is not an isolated point in $I(\theta)$; $\theta' + \varepsilon \in I(\theta)$, for any $\varepsilon > 0$. For a large enough K , we have $b_k(x) = b_k(y)$ for all $x, y \in I(\theta) - \{\theta'\}$ and $b_k(\theta') < b_k(\theta)$. If $b_k(\theta) < \theta'$, the designer can increase the revenue similarly, and it leads to a contradiction to that $b_k(\cdot)$ is the objective maximizer. If $b_k(\theta) = \theta'$, there exists a small $\varepsilon > 0$ such that $\theta' + \varepsilon$ would deviate to $b_k(\theta')$ and it leads to a contradiction to that $b_k(\cdot)$ is in an equilibrium.

Claim 5. $\mu_k(\{x|b_k(x) < b_k(\theta)\}) \rightarrow \mu(\{x|b(x) < b(\theta)\})$.

Proof of Claim 5: Notice that if $b(x) < b(\theta)$, then there exists K such that $b_k(x) < b_k(\theta)$ for all $k > K$, thus, $\{x|b(x) < b(\theta)\} \subseteq \liminf_k \{x|b_k(x) < b_k(\theta)\}$ and

$$\liminf_k \mu_k(\{x|b_k(x) < b_k(\theta)\}) \geq \mu(\{x|b(x) < b(\theta)\}).$$

Suppose that $\theta' \in \limsup_k \{x|b_k(x) < b_k(\theta)\}$ but $b(\theta') = b(\theta)$. By Claim 3, $\theta' \in \liminf_k I_k(\theta)$ or $\mu(I(\theta)) = 0$. If $\theta' \in \liminf_k I_k(\theta)$, then it is a contradiction to $\theta' \in \limsup_k \{x|b_k(x) < b_k(\theta)\}$, thus, $\liminf_k \mu_k(\{x|b_k(x) < b_k(\theta)\}) \geq \mu(\{x|b(x) < b(\theta)\}) \geq \limsup_k \mu_k(\{x|b_k(x) < b_k(\theta)\})$.

Suppose that $\mu(I(\theta)) = 0$. By noticing $\mu(\{x|b(x) \leq b(\theta)\}) = \mu(\{x|b(x) < b(\theta)\})$, one can easily see that

$$\limsup_k \mu_k(\{x|b_k(x) < b_k(\theta)\}) \leq \mu(\{x|b(x) < b(\theta)\}) \leq \liminf_k \mu_k(\{x|b_k(x) < b_k(\theta)\}).$$

Claim 4 and 5 imply that $b(\cdot)$ is in an equilibrium in the limit.

It remains to show that the equilibrium $b(\cdot)$ of the first-price auction with B implements an optimal social rule. Let $M(\mu)$ denotes the set of feasible, individually rational, and incentive compatible social rules with respect to μ . For $U(y, \mu)$ and $U(y_k, \mu_k)$, define

$$f(y) = \begin{cases} U(y, \mu), & y \in M(\mu), \\ -\infty, & y \notin M(\mu). \end{cases}$$

and

$$f^k(y_k) = \begin{cases} U(y_k, \mu_k), & y_k \in M(\mu_k), \\ -\infty, & y_k \notin M(\mu_k). \end{cases}$$

For any $y_k \rightarrow y$, $\limsup_k f^k(y_k)$ is either $f(y)$ or $-\infty$, thus, $\limsup_k f^k(y_k) \leq \lim_k f^k(y_k)$.

To show that $\liminf_k f^k(y_k) \geq f(y)$ for some $y_k \rightarrow y$, let us select y_k that has the reduced form of y on Θ_k . Notice that y_k is incentive compatible on Θ_k . By the continuity of U , $f^k(Q, T) = U(Q, T, \mu_k) \rightarrow U(Q, T, \mu) = f(Q, T)$ as $k \rightarrow \infty$. Therefore, f^k hypo-converges to f .

Since the social rule implemented by $b_k(\cdot)$ is optimal at each step k , any social rule implemented by a limit point of $b_k(\cdot)$ is optimal by Theorem 7.33 in Rockafellar and Wets (1998). Because we already established that b is a limit point of b_k , it is optimal.

Proof of Theorem 4

If the bid space is finite, the existence of a pure-strategy symmetric equilibrium is implied by Philip J. Reny (2011). Because \mathcal{B} is compact, it can be approximated by finite subsets \mathcal{B}_k so that for any $b \in \mathcal{B}$, $(b - \frac{1}{k}, b + \frac{1}{k}) \cap \mathcal{B}_k \neq \emptyset$ for any $k \in \mathbb{N}$. For any $k \in \mathbb{N}$, there exists a symmetric and monotonic pure-strategy equilibrium of the first-price auction with bid space \mathcal{B}_k . Let us select one such equilibrium strategy and denote it $b_k(\cdot)$. Because \mathcal{B} is compact, we can ensure that each \mathcal{B}_k includes $\min(\mathcal{B})$ and $\max(\mathcal{B})$. We can further assume that $\mathcal{B}_k \subseteq \mathcal{B}_{k+1}$. We then have a sequence of monotonic functions $b_k(\cdot)$ which admits a point-wise convergent subsequence by Helly's selection theorem. For notational convenience, we can assume that $(b_k)_{k \in \mathbb{N}}$ itself is this convergent subsequence. Let us denote its limit by $b(\cdot)$. We claim that the limit constitutes a symmetric equilibrium in the first-price auction under \mathcal{B} . For any $k \in \mathbb{N}$, the probability of winning for type θ only depends on the $\mu(\{x | b_k(x) < b_k(\theta)\})$ and $\mu(\{x | b_k(x) = b_k(\theta)\})$, and it can be expressed as

$$(\mu(\{x | b_k(x) < b_k(\theta)\}))^{N-1} + \sum_{n=1}^{N-1} \frac{1}{n+1} \binom{N-1}{n} (\mu(\{x | b_k(x) < b_k(\theta)\}))^{N-1-n} (\mu(\{x | b_k(x) = b_k(\theta)\}))^n.$$

Let us denote $\{x | b_k(x) = b_k(\theta)\}$ by $I_k(\theta)$ and $\{x | b(x) = b(\theta)\}$ by $I(\theta)$. We use the following claims to prove that $b(\cdot)$ is in equilibrium.

Claim 1. $\limsup_k I_k(\theta) \subseteq I(\theta)$.

Proof. If $x \in \limsup_k I_k(\theta)$, then $b_k(x) = b_k(\theta)$ infinitely often. Since $b_k(\cdot)$ converges to $b(\cdot)$, $b(x) = b(\theta)$, thus $x \in I(\theta)$.

Claim 2. If $\mu(I(\theta)) > 0$, then:

either $I(\theta) \subseteq \liminf_k I_k(\theta)$,

or $b(x) = x$ for some $x \in I(\theta)$ and $I(\theta) - \{x\} \subseteq \liminf_k I_k(\theta)$.

Proof. Suppose that there exists $\theta' \in I(\theta) - \liminf_k I_k(\theta)$. Notice that $\theta' \neq \theta$ because $\liminf_k I_k(\theta)$ includes θ . For any K , there exists $k > K$ such that $b_k(\theta') \neq b_k(\theta)$. The

monotonicity of b_k allows us to pick $x, y \in I(\theta)$ such that for any K , there exists $k > K$ such that $b_k(x) < b_k(y)$ and $\mu([x, y]) > 0$. Because $b_k(\cdot)$ is in a symmetric equilibrium, the following must hold:

$$[P_k(b_k(y)) - P_k(b_k(x))](x - b_k(x)) \leq P_k(b_k(y))(b_k(y) - b_k(x)),$$

where $P_k(b)$ represents the winning probability from placing a bid b assuming all the other bidders follow the bidding strategy $b_k(\cdot)$.

Notice that the right hand side of the inequality is arbitrarily close to 0, but the limit of left hand side is bounded below by $\frac{1}{N}\mu([x, y])^{N-1}(x - b(x))$ because the gain in the winning probability by changing a bid from $b_k(x)$ to $b_k(y)$ is at least $\frac{1}{N}\mu([x, y])^{N-1}$. Therefore, if $b(x) < x$, it leads to a contradiction because the type x would deviate to $b_k(y)$ for some k .

Claim 3. $\mu(I_k(\theta)) \rightarrow \mu(I(\theta))$.

Proof. For $I(\theta)$ such that $\mu(I(\theta)) > 0$, we established $\limsup_k I_k(\theta) \subseteq I(\theta) \subseteq \liminf_k I_k(\theta)$ with an exception that might arise for a single type $x \in I(\theta)$ such that $x = b(\theta)$, which can be ignored by the atom-less assumption. Therefore, we establish that $\lim_k \mu(I_k(\theta)) = \mu(I(\theta))$ by applying Fatou's lemma. For $I(\theta)$ such that $\mu(I(\theta)) = 0$, we still have $\limsup_k I_k(\theta) \subseteq I(\theta)$. Because $\mu(I_k(\theta)) \geq 0$ for all k and $\mu(I(\theta)) = 0$, $\lim_k \mu(I_k(\theta)) = 0$.

Claim 4. For any $\theta \in \Theta$, $\mu(\{x|b_k(x) < b_k(\theta)\}) \rightarrow \mu(\{x|b(x) < b(\theta)\})$.

Proof. We want to show that $\limsup_k \{x|b_k(x) < b_k(\theta)\} \subseteq \{x|b(x) < b(\theta)\}$ and $\{x|b(x) < b(\theta)\} \subseteq \liminf_k \{x|b_k(x) < b_k(\theta)\}$ μ -almost everywhere. To show the first, suppose that there exists $\theta' \in \limsup_k \{x|b_k(x) < b_k(\theta)\}$ but $b(\theta') \geq b(\theta)$. Because $b_k(\cdot)$ converges to $b(\cdot)$, it must be $b(\theta') = b(\theta)$. By claim 2, either $\mu(I(\theta)) = 0$ or $\theta' \in \liminf_k I_k(\theta)$. If $\mu(I(\theta)) = 0$, then $\limsup_k \{x|b_k(x) < b_k(\theta)\} \subseteq \{x|b(x) < b(\theta)\}$ μ -almost everywhere. If $\theta' \in \liminf_k I_k(\theta)$, then it leads to a contradiction to the assumption that $\theta' \in \limsup_k \{x|b_k(x) < b_k(\theta)\}$. It remains to show that $\{x|b(x) < b(\theta)\} \subseteq \liminf_k \{x|b_k(x) < b_k(\theta)\}$ μ -almost everywhere. If $b(\theta') < b(\theta)$, then there exists K such that $b_k(\theta') < b_k(\theta)$ for all $k \geq K$, thus $\theta' \in \liminf_k \{x|b_k(x) < b_k(\theta)\}$. Therefore, $\mu(\{x|b_k(x) < b_k(\theta)\}) \rightarrow \mu(\{x|b(x) < b(\theta)\})$.

Claim 5. If $b \in \mathcal{B} - b(\Theta)$ then $\mu(\{x|b_k(x) < b\}) \rightarrow \mu(\{x|b(x) < b\})$.

Proof. Pick an arbitrary $\theta \in \limsup_k \{x|b_k(x) < b\}$. Then, $b_k(\theta) < b$ infinitely often and $b_k(\theta) \rightarrow b(\theta) \neq b$. This implies that $b(\theta) < b$, thus, $\limsup_k \{x|b_k(x) < b\} \subseteq \{x|b(x) < b\}$. To see $\{x|b(x) < b\} \subseteq \liminf_k \{x|b_k(x) < b\}$, pick an arbitrary θ such that $b(\theta) < b$. Because $b_k(\theta)$ converges to $b(\theta)$, for any $\varepsilon > 0$, there exists K such that $b_k(\theta) < b - \varepsilon$ for all $k > K$. Therefore, $\theta \in \liminf_k \{x|b_k(x) < b\}$. Applying Fatou's lemma completes the proof.

Claim 3 and 4 establish that there is no profitable deviation to a bid $b \in b(\Theta)$. Consider a deviation to a bid $b \in \mathcal{B} - b(\Theta)$. Because $P_k(b)$ converges to $P(b)$ by Claim 5, if b were a profitable deviation in the limit and belonged to some \mathcal{B}_ℓ then the monotonicity of \mathcal{B}_k ensures

that it would also be a profitable deviation at some step $k \geq \ell$. It remains to consider the case in which $b \notin \cup_{k \in \mathbb{N}} \mathcal{B}_k$. If b were a profitable deviation and $\theta - b \leq 0$, so were deviating to $b - \varepsilon$ for any $\varepsilon \geq 0$. Because $\cup_{k \in \mathbb{N}} \mathcal{B}_k$ is dense in \mathcal{B} , there exists $\varepsilon \geq 0$ such that $b - \varepsilon \in \cup_{k \in \mathbb{N}} \mathcal{B}_k$, and there would be a profitable deviation to $b - \varepsilon$ at some step k . If b were a profitable deviation and $\theta - b > 0$, then there exists small $\varepsilon > 0$ such that deviating to $b + \varepsilon'$ is also a profitable deviation for any $\varepsilon' \in [0, \varepsilon]$. Because $\cup_{k \in \mathbb{N}} \mathcal{B}_k$ is dense in \mathcal{B} , there exists $b' \in \cup_{k \in \mathbb{N}} \mathcal{B}_k$ such that $b' \in [b, b + \varepsilon]$ and there would be a profitable deviation to b' at some step k .

Proof of Lemma 2

Given a symmetric profile of bidding strategies $b(\cdot)$, regardless of the restrictions on the bid space, we can express the winning probability of a bidder by placing a bid $b \in \mathbb{R}$ as

$$P(b) = (\mu(\{\theta | b > b(\theta)\}))^{N-1} + \sum_{n=1}^{N-1} \frac{1}{n+1} \binom{N-1}{n} (\mu(\{\theta | b > b(\theta)\}))^{N-1-n} (\mu(\{\theta | b = b(\theta)\}))^n,$$

where we use two conventions: (i) $b(\theta) = -\infty$ if type θ drops and does not submit a bid, and (ii) we resolve $(\mu(\{\theta | b > b(\theta)\}))^0$ to equal 1 when $\mu(\{\theta | b > b(\theta)\}) = 0$. The win probability $P(\cdot)$ is monotone in b and, for $b > \inf(\mathcal{B})$, it is discontinuous at b if and only if $\mu(\{\theta | b = b(\theta)\}) > 0$. Suppose that $b(\cdot)$ is in equilibrium under \mathcal{B} , but not under $cl(\mathcal{B})$. Then, there exists a type $\theta \in \Theta$ and $b \in cl(\mathcal{B}) - \mathcal{B}$ such that $P(b)(\theta - b) > P(b(\theta))(\theta - b(\theta))$. Since $b \in cl(\mathcal{B}) - \mathcal{B}$, there exists a sequence $(b_k)_{k \in \mathbb{N}}$ that converges to b with each $b_k \in \mathcal{B}$ for all k . Consider a case where $b > \inf(\mathcal{B})$. Because $\mu(\{\theta | b = b(\theta)\}) = 0$ as b is not a valid bid under \mathcal{B} , thus, $P(\cdot)$ is continuous at b , there exists k such that $P(b_k)(\theta - b_k) > P(b(\theta))(\theta - b(\theta))$, which is a contradiction because $b(\cdot)$ is an equilibrium under \mathcal{B} . In the remaining case, $b = \inf(\mathcal{B})$; notice that $b_k \geq b$ for all k in the convergent sequence (b_k) with $b_k \in \mathcal{B}$. Thus, $P(b_k) \geq P(b)$ for all k , and b_k can be arbitrarily close to b , thus, $P(b_k)(\theta - b_k) > P(b(\theta))(\theta - b(\theta))$ for some k , which leads to the same contradiction.

Proof of the Claim in Example 1

Take an arbitrary bid space $\hat{\mathcal{B}}$ and a pure strategy equilibrium bidding strategy $\hat{b}(\cdot)$. Since $\hat{b}(\cdot)$ is monotone in type, there are at most countable points of discontinuity, and we denote the set of types where $\hat{b}(\cdot)$ is discontinuous by $\Theta(\hat{b}) = \{\theta_0, \theta_1, \dots\}$, where $\theta_k < \theta_{k+1}$ for all $k = 0, 1, \dots$. Denote the probability of the winning by placing a bid b by $\hat{Q}(b)$ assuming that all the other bidders follow the equilibrium bidding strategy $\hat{b}(\cdot)$. The expected payoff of a bidder with type θ when the bidder chooses b can be expressed as $\hat{u}(\theta, b) = \hat{Q}(b)\theta - \hat{Q}(b)b$.

For any $\theta \notin \Theta(\hat{b})$ and $\theta > r$, the following envelope condition is necessary for $\hat{b}(\cdot)$ to be an equilibrium:

$$\hat{b}(\theta) = \theta - \frac{\int_r^\theta \hat{Q}(x)dx}{\hat{Q}(\theta)}. \quad (7)$$

Notice that $\hat{b}(\cdot)$ is either strictly increasing or completely flat on (θ_{k-1}, θ_k) for any $k \in \mathbb{N}$ almost everywhere. Otherwise, there is a jump in the winning probability at some type $\theta \in (\theta_{k-1}, \theta_k)$, while the bid does not change as much by the continuity of $\hat{b}(\cdot)$ on (θ_{k-1}, θ_k) , and this breaks the equilibrium. If $\hat{b}(\cdot)$ is strictly increasing on (θ_{k-1}, θ_k) , then $\hat{Q}(\cdot) = Q^*(\cdot)$ on (θ_{k-1}, θ_k) by the rule of the first-price auction, where $Q^*(\cdot)$ denotes the efficient interim allocation. If $\hat{b}(\cdot)$ is flat on (θ_{k-1}, θ_k) , the following equalities hold:

$$\int_{\theta_{k-1}}^{\theta_k} \hat{Q}(x)dx = (\theta_k - \theta_{k-1}) \left[\theta_{k-1} + \frac{1}{2}(\theta_k - \theta_{k-1}) \right] = \frac{1}{2}\theta_k^2 - \frac{1}{2}\theta_{k-1}^2 = \int_{\theta_{k-1}}^{\theta_k} xdx = \int_{\theta_{k-1}}^{\theta_k} Q^*(x)dx.$$

Using the observation above, we can characterize the equilibrium $\hat{b}(\cdot)$ on (θ_{k-1}, θ_k) as follows:

If $\hat{b}(\cdot)$ is flat on (θ_{k-1}, θ_k) , $\hat{Q}(\theta) = \theta_{k-1} + \frac{1}{2}(\theta_k - \theta_{k-1})$ and $\int_r^\theta \hat{Q}(x)dx = \sum_{\ell < k} \int_{\theta_{\ell-1}}^{\theta_\ell} Q^*(x)dx + (\theta - \theta_{k-1}) \left[\theta_{k-1} + \frac{1}{2}(\theta_k - \theta_{k-1}) \right]$.

If $\hat{b}(\cdot)$ is strictly increasing on (θ_{k-1}, θ_k) , $\hat{Q}(\theta) = Q^*(\theta)$ and $\int_r^\theta \hat{Q}(x)dx = \sum_{\ell < k} \int_{\theta_{\ell-1}}^{\theta_\ell} Q^*(x)dx + \int_{\theta_{k-1}}^\theta Q^*(x)dx$.

Using that $Q^*(x) = x$, the following closed form can be derived:

$$\hat{b}(\theta) = \begin{cases} \frac{\theta_{k-1}\theta_k + r^2}{\theta_{k-1} + \theta_k} & \text{if } \hat{b} \text{ is flat on } (\theta_{k-1}, \theta_k), \\ \frac{1}{2}\theta + \frac{1}{2}r^2 & \text{otherwise.} \end{cases}$$

As the bids that are not selected in the equilibrium do not affect the objective, we can assume that $\hat{b}(1) = \bar{b}$. Notice that \bar{b} is either $\frac{\theta^* + r^2}{\theta^* + 1}$ or $\frac{1}{2} + \frac{1}{2}r^2$, where θ^* is the infimum of types that chooses \bar{b} .

With bid space $[r, \bar{b}]$, if the bidding cap \bar{b} is binding, then the following bidding strategy $b(\cdot)$ constitutes a symmetric equilibrium: $b(\theta) = \emptyset$ if $\theta < r$, $b(\theta) = \frac{1}{2}\theta + \frac{1}{2}r^2$ if $r \leq \theta < \frac{\bar{b} - r^2}{1 - \bar{b}}$, and $b(\theta) = \bar{b}$ otherwise. All other symmetric equilibria might coincide with this one except that the cutoff types r and $\frac{\bar{b} - r^2}{1 - \bar{b}}$, who are indifferent between two bids, might select another one. This equilibrium multiplicity is non-essential as all equilibria lead to the same payoffs for every type and same expected objective for the designer (as there is no atom and changes at countable points neither affect the objective of the designer nor the incentive of bidders). The equilibrium we focus on is the unique symmetric equilibrium in which the bids are right-continuous in type. The monotonicity of bids similarly implies that we may focus on a

right-continuous equilibrium bidding strategy $\hat{b}(\cdot)$ when the bid space is $\hat{\mathcal{B}}$. Let the resulting equilibrium expected payoff be $\hat{u}(\cdot)$ and the interim winning probability $\hat{Q}(\cdot)$.

Let us show that $\hat{Q}(1) \geq Q(1)$, where $Q(\cdot)$ denotes the interim winning probability in the equilibrium under $\mathcal{B} = [r, \hat{b}]$. Whenever there is a jump in \hat{b} at θ , the type θ is indifferent between $\hat{b}(\theta)$ and $\lim_{x \rightarrow \theta^-} \hat{b}(x)$ since there is no atom. Otherwise, types within an ϵ -ball around θ would deviate. Another observation that can be made is that if \hat{b} is not flat on the left side of θ , then $\lim_{x \rightarrow \theta^-} \hat{b}(x) = \frac{1}{2}\theta + \frac{1}{2\theta}r^2$, which is an implication from the envelope theorem. Let us denote the set of types such that $\hat{b}(\cdot)$ is discontinuous by $\Theta(\hat{b}) = \{\theta_0, \theta_1, \dots\}$. We show that for any $\theta_k \in \Theta(\hat{b})$, $\hat{u}(\theta_k) = u(\theta_k)$ by induction, where $u(\cdot)$ denotes the expected utility of θ_k in the equilibrium with $\mathcal{B} = [r, \hat{b}]$. If $\lim_{x \rightarrow \theta_0^-} \hat{b}(x) = r$, the type θ_0 is indifferent between bidding r and $\hat{b}(\theta_0)$. The probability of winning by bidding r is $\frac{1}{2}(\theta_0 - r) + r$ and the expected utility $\hat{u}(\theta_0) = (\frac{1}{2}(\theta_0 - r) + r) [\theta_0 - r] = \frac{1}{2}\theta_0^2 - \frac{1}{2}r^2$. For the inductive step, suppose that $\hat{u}(\theta_k) = \frac{1}{2}\theta_k^2 - \frac{1}{2}r^2$. If $\hat{b}(\theta_k) < \lim_{x \rightarrow \theta_{k+1}^-} \hat{b}(x)$ and the type θ_{k+1} is indifferent between bidding $\frac{1}{2}\theta_{k+1} + \frac{1}{2\theta_{k+1}}r^2$ and $\hat{b}(\theta_{k+1})$, then $\hat{u}(\theta_{k+1}) = \theta_{k+1} \left[\theta_{k+1} - \frac{1}{2}\theta_{k+1} - \frac{1}{2\theta_{k+1}}r^2 \right] = \frac{1}{2}\theta_{k+1}^2 - \frac{1}{2}r^2$. If $\hat{b}(\cdot)$ is flat from θ_k to θ_{k+1} , the expected utility of the type θ_{k+1} is the sum of $\hat{u}(\theta_k)$ and $(\theta_k + \frac{1}{2}(\theta_{k+1} - \theta_k)) [\theta_{k+1} - \theta_k]$, which is $\frac{1}{2}\theta_k^2 - \frac{1}{2}r^2 + \frac{1}{2}\theta_{k+1}^2 - \frac{1}{2}\theta_k^2 = \frac{1}{2}\theta_{k+1}^2 - \frac{1}{2}r^2$.

Let us denote the minimum type that submit \bar{b} by θ^* . If $\Theta(\hat{b}) = \{\theta_0, \theta_1, \dots\}$ is finite, we have already established that $\hat{u}(\theta^*) = \frac{1}{2}\theta^{*2} - \frac{1}{2}r^2$. If $\Theta(\hat{b})$ is infinite, by the monotone convergence theorem, the sequence (θ_k) converges to $\sup(\Theta(\hat{b}))$. If $\theta^* > \sup(\Theta(\hat{b}))$, then there is no jump at θ^* , and it implies that $\theta^* = 1$ and $\hat{Q}(1) = 1$. If $\theta^* = \sup(\Theta(\hat{b}))$, then $\hat{u}(\theta^*) = \lim_{k \rightarrow \infty} \hat{u}(\theta_k) = \frac{1}{2}\theta^{*2} - \frac{1}{2}r^2$ since \hat{u} is continuous as there is no atom. Moreover, $\hat{u}(1) = \hat{u}(\theta^*) + (\theta^* + \frac{1}{2}(1 - \theta^*)) [1 - \theta^*] = \frac{1}{2} - \frac{1}{2}r^2$ which coincides with the expected utility $u(1)$ in the equilibrium with $\mathcal{B} = [r, \hat{b}]$. This implies that the last cutoff type θ^* is $\frac{\bar{b}-r^2}{1-\bar{b}}$ and $Q(1) = \hat{Q}(1)$. Therefore, if the minimum bid allowed and the maximum bid allowed are given, the design problem reduces to the revenue maximization with two fixed end points in the bid space.

In order to compare the revenue in between $\hat{b}(\cdot)$ and $b(\cdot)$, notice that $\hat{b}(\theta) = b(\theta)$ if $\hat{b}(\cdot)$ is strictly increasing at θ , therefore it is enough to compare the expected payment from types on the flat bidding region. Suppose that $\hat{b}(\cdot)$ is flat on (θ_{k-1}, θ_k) . The expected payment from type $\theta \in (\theta_{k-1}, \theta_k)$ in the equilibrium $\hat{b}(\cdot)$ can be express as

$$\hat{T}(\theta) = \frac{\theta_{k-1}\theta_k + r^2}{\theta_{k-1} + \theta_k} \hat{Q}(\theta) = \frac{1}{2}\theta_{k-1}\theta_k + \frac{1}{2}r^2,$$

Thus, the conditional expected revenue on (θ_{k-1}, θ_k) in the equilibrium $\hat{b}(\cdot)$ is:

$$\int_{\theta_{k-1}}^{\theta_k} \hat{T}(\theta) d\theta = \frac{1}{2} \theta_{k-1} \theta_k (\theta_k - \theta_{k-1}) + \frac{1}{2} r^2 (\theta_k - \theta_{k-1}).$$

The expected payment from type $\theta \in (\theta_{k-1}, \theta_k)$ in the equilibrium $b(\cdot)$ can be express as

$$T(\theta) = \left[\frac{1}{2} \theta + \frac{1}{2} r^2 \right] Q(\theta) = \frac{1}{2} \theta^2 + \frac{1}{2} r^2,$$

The conditional expected revenue on (θ_{k-1}, θ_k) in the equilibrium $b(\cdot)$ is:

$$\int_{\theta_{k-1}}^{\theta_k} T(\theta) d\theta = \frac{1}{6} (\theta_k^3 - \theta_{k-1}^3) + \frac{1}{2} r^2 (\theta_k - \theta_{k-1}).$$

It can be verified that $\int_{\theta_{k-1}}^{\theta_k} T(\theta) d\theta - \int_{\theta_{k-1}}^{\theta_k} \hat{T}(\theta) d\theta = \frac{1}{6} (\theta_k - \theta_{k-1})^3$, thus $b(\cdot)$ dominates $\hat{b}(\cdot)$ in terms of maximizing the expected revenue.

Because $b(\cdot)$ dominates $\hat{b}(\cdot)$ in terms of maximizing the expected revenue and weakly dominates in terms of minimizing the disparity $Q(1) - Q(0)$, it is enough to consider $\mathcal{B} = [r, \bar{b}]$. This proves the claim.

Proof of Lemma 4

Proof. (i) \iff (iii): For this proof, let us introduce a notation for CDF of π , $F(\theta) = \sum_{x < \theta} \pi(x)$. Notice that the set \mathcal{Q} in (6) is characterized by $2T$ linear constraints as follows.

$$\mathcal{Q} = \left\{ Q : \Theta \rightarrow [0, 1] \mid N \sum_{s=t}^T Q(\theta^s) \pi(\theta^s) \leq 1 - F(\theta^t)^N, Q(\theta^t) \geq Q(\theta^{t-1}), \right. \\ \left. \text{for all } t = 1, \dots, T \right\}.$$

Let us denote $N \sum_{s=t}^T Q(\theta^s) \pi(\theta^s) \leq 1 - F(\theta^t)^N$ by “ t -th feasibility constraint”. Denote $Q(\theta^t) \geq Q(\theta^{t-1})$ by “ t -th monotonicity constraint” for $t > 1$. First of all, notice that any T constraints in \mathcal{Q} are linearly independent. Suppose that Q is an extreme point of \mathcal{Q} and denote the partition that characterizes Q by $\{\Theta_1, \dots, \Theta_J\}$. $T - J$ number of monotonicity constraints are binding. First, we show that there exists $\theta^{t_j} \in \Theta_j$ such that t_j -th feasibility constraint is binding, for each $j > 1$. Suppose not. Then there exists Θ_j such that no

feasibility constraints are binding for all $\theta \in \Theta_j$. Define Q' and Q'' as follows.

$$Q'(\theta) = \begin{cases} Q(\theta) & \text{if } \theta \notin \Theta_j, \\ Q(\theta) + \epsilon & \text{if } \theta \in \Theta_j. \end{cases}$$

$$Q''(\theta) = \begin{cases} Q(\theta) & \text{if } \theta \notin \Theta_j, \\ Q(\theta) - \epsilon & \text{if } \theta \in \Theta_j. \end{cases}$$

By taking ϵ small enough, both Q' and Q'' are in \mathcal{Q} . Moreover, $0.5Q' + 0.5Q'' = Q$. This is contradiction to that Q is an extreme point. Notice that if $Q(\theta_1)$ is 0, Q'' is not well-defined, thus, the feasibility constraint does not need to be binding for $j = 1$. However, if $Q(\Theta_1) > 0$, the feasibility constraint for $j =$ must be binding by the same argument. As a next step, we show that $\theta^{t_j} = \min \Theta_j$ for each $j > 1$. We will prove this by induction. Before proceeding further with the proof, the following variation of geometric summation formula will be useful in reading the proof.

$$\frac{x^n - y^n}{x - y} = \sum_{s=0}^{n-1} x^{n-1-s} y^s.$$

Consider Θ_J and θ^{t_J} . Since t_J -th feasibility constraint is binding,

$$NQ(\Theta_J)(1 - F(\theta^{t_J})) = 1 - F(\theta^{t_J})^N,$$

and

$$NQ(\Theta_J) = \frac{1 - F(\theta^{t_J})^N}{1 - F(\theta^{t_J})} = \sum_{s=0}^{N-1} F(\theta^{t_J})^s.$$

Suppose that there exists $\theta \in \Theta_J$ such that $\theta < \theta^{t_J}$. The feasibility constraint evaluated at θ is

$$NQ(\Theta_J)(1 - F(\theta)) \leq 1 - F(\theta)^N,$$

and

$$NQ(\Theta_J) \leq \frac{1 - F(\theta)^N}{1 - F(\theta)} = \sum_{s=0}^{N-1} F(\theta)^s. \quad (8)$$

This is a contradiction because $F(\theta^{t_j}) > F(\theta)$. Now, suppose that $\theta^{t_{j+1}} = \min \Theta_{j+1}$ and consider Θ_j and θ^{t_j} . If $\theta^{t_j} > \min \Theta_j$, there exist $\theta \in \Theta_j$ such that $\theta < \theta^{t_j}$. Since t_j -th and t_{j+1} -th feasibility constraints are binding,

$$\begin{aligned} NQ(\Theta_j)(F(\theta^{t_{j+1}}) - F(\theta^{t_j})) + N \sum_{x \geq \theta^{t_{j+1}}} Q(x)\pi(x) &= 1 - F(\theta^{t_j})^N, \\ N \sum_{x \geq \theta^{t_{j+1}}} Q(x)\pi(x) &= 1 - F(\theta^{t_{j+1}})^N. \end{aligned}$$

By taking the difference,

$$NQ(\Theta_j) = \frac{F(\theta^{t_{j+1}})^N - F(\theta^{t_j})^N}{F(\theta^{t_{j+1}}) - F(\theta^{t_j})} = \sum_{s=0}^{N-1} F(\theta^{t_{j+1}})^{N-1-s} F(\theta^{t_j})^s.$$

By evaluating the feasibility constraint at θ , the same derivation leads to

$$NQ(\Theta_j) \leq \sum_{s=0}^{N-1} F(\theta^{t_{j+1}})^{N-1-s} F(\theta)^s,$$

which is a contradiction since $F(\theta^{t_j}) > F(\theta)$. The above step also shows that

$$\begin{aligned} NQ(\Theta_j) &= \frac{F(\theta^{t_{j+1}})^N - F(\theta^{t_j})^N}{F(\theta^{t_{j+1}}) - F(\theta^{t_j})} = \frac{F(\theta^{t_{j+1}})^N - F(\theta^{t_j})^N}{\pi(\Theta_j)}, \\ &= \frac{(\sum_{k=1}^j \pi(\Theta_k))^N - (\sum_{k=1}^{j-1} \pi(\Theta_k))^N}{\pi(\Theta_j)}, \text{ for } j > 1. \end{aligned}$$

For $j = 1$, if $Q(\Theta_1) > 0$, then the binding feasibility constraint leads to

$$NQ(\Theta_1) = \frac{\pi(\Theta_1)^N}{\pi(\Theta_1)} = \pi(\Theta_1)^{N-1}.$$

For $j = 1$, if $Q(\theta_1) = 0$, then $Q(\theta_1) \geq 0$ constraint must be binding instead of the feasibility constraint since we need T number of binding constraints for Q to be an extreme point. For converse, if Q takes the form in (iii), it is easy to see that T number of constraints are binding at Q .

(ii) \iff (iii):

$$\begin{aligned} Q(\Theta_j) &= \sum_{n=1}^N \frac{1}{n} \binom{N-1}{n-1} \pi(\Theta_j)^{n-1} \left[\sum_{l=1}^{j-1} \pi(\Theta_l) \right]^{N-n} \\ &= \frac{1}{N} \sum_{n=1}^N \binom{N}{n} \pi(\Theta_j)^{n-1} \left[\sum_{l=1}^{j-1} \pi(\Theta_l) \right]^{N-n}, \end{aligned}$$

because $\frac{N}{n} \binom{N-1}{n-1} = \binom{N}{n}$. Applying the binomial formula,

$$\begin{aligned} & \frac{1}{N} \sum_{n=1}^N \binom{N}{n} \pi(\Theta_j)^{n-1} \left[\sum_{l=1}^{j-1} \pi(\Theta_l) \right]^{N-n} \\ &= \frac{1}{N\pi(\Theta_j)} \sum_{n=1}^N \binom{N}{n} \pi(\Theta_j)^n \left[\sum_{l=1}^{j-1} \pi(\Theta_l) \right]^{N-n} \\ &= \frac{1}{N\pi(\Theta_j)} \left[\sum_{n=0}^N \binom{N}{n} \pi(\Theta_j)^n \left[\sum_{l=1}^{j-1} \pi(\Theta_l) \right]^{N-n} - \left[\sum_{l=1}^{j-1} \pi(\Theta_l) \right]^N \right] \end{aligned} \quad (9)$$

$$= \frac{1}{N\pi(\Theta_j)} \left[\left[\sum_{l=1}^{j-1} \pi(\Theta_l) + \pi(\theta_j) \right]^N - \left[\sum_{l=1}^{j-1} \pi(\Theta_l) \right]^N \right] \quad (10)$$

$$= \frac{1}{N\pi(\Theta_j)} \left[\left[\sum_{l=1}^j \pi(\Theta_l) \right]^N - \left[\sum_{l=1}^{j-1} \pi(\Theta_l) \right]^N \right]. \quad (11)$$

The binomial formula is used from (9) to (10). If $j = 1$, (11) becomes $\frac{\pi(\Theta_1)^{N-1}}{N}$. If the seller keeps the item when the all bidders' types are in Θ_1 , then $Q(\Theta_1)$ is 0. To show the converse, we established that if Q satisfies (49) and (50), then $Q(\Theta_{j+1}) > Q(\Theta_j)$ because

$$NQ(\Theta_j) = \frac{F(\theta^{t_{j+1}})^N - F(\theta^{t_j})^N}{F(\theta^{t_{j+1}}) - F(\theta^{t_j})} = \sum_{s=0}^{N-1} F(\theta^{t_{j+1}})^{N-1-s} F(\theta^{t_j})^s$$

Thus, the hierarchical allocation rule with a partition $\{\Theta_1, \dots, \Theta_J\}$ implements Q . \square

Proof of Lemma 5

Proof. Denote the coefficient of Q_i^K by α_k . It is enough to show that the following system is feasible, where $T_i^k(\theta^t)$'s are variables.

$$\begin{aligned} Q_i^k(\theta^t)\theta^t - T_i^k(\theta^t) &\geq Q_i^k(\theta^{t-1})\theta^t - T_i^k(\theta^{t-1}), \text{ for } t = 2, \dots, T \text{ and all } k, \\ Q_i^k(\theta^{t-1})\theta^{t-1} - T_i^k(\theta^{t-1}) &\geq Q_i^k(\theta^t)\theta^{t-1} - T_i^k(\theta^t), \text{ for } t = 2, \dots, T \text{ and all } k, \\ Q_i^k(\theta^1)\theta^1 - T_i^k(\theta^1) &\geq 0 \text{ for all } k, \\ \sum_{k=1}^K \alpha_k T_i^k(\theta^t) &= T_i(\theta^t), \text{ for all } t = 1, \dots, T. \end{aligned}$$

Let $\Delta Q_i^k(\theta^t) = Q_i^k(\theta^t) - Q_i^k(\theta^{t-1})$, then the system of inequalities are

$$\begin{aligned} T_i^k(\theta^t) - T_i^k(\theta^{t-1}) &\leq \Delta Q_i^k(\theta^t)\theta^t, \text{ for } t = 1, \dots, T \text{ and all } k, \\ -T_i^k(\theta^t) + T_i^k(\theta^{t-1}) &\leq -\Delta Q_i^k(\theta^t)\theta^{t-1}, \text{ for } t = 2, \dots, T \text{ and all } k, \\ \sum_{k=1}^K \alpha_k T_i^k(\theta^t) &= T_i(\theta^t), \text{ for all } t = 1, \dots, T. \end{aligned}$$

Denote the dual variables corresponding to $\Delta Q_i^k(\theta^t)\theta^s$ by $z_{t,s}^k$ and the dual variables corresponding to $T_i(\theta^t)$ by y_t . Then, the Farkas alternative of the original system is

$$-z_{t+1,t+1}^k + z_{t+1,t}^k + z_{t,t}^k - z_{t,t-1}^k + \alpha_k y_t = 0, \text{ for all } t \text{ and } k, \quad (12)$$

$$\sum_{t=1}^T \sum_{k=1}^K [z_{t,t}^k \Delta Q_i^k(\theta^t)\theta_t - z_{t,t-1}^k \Delta Q_i^k(\theta^t)\theta^{t-1}] + \sum_{t=1}^T y_t T_i(\theta^t) < 0. \quad (13)$$

By rearranging (12) and (13),

$$z_{t,t}^k - z_{t,t-1}^k = -\alpha_k \sum_{s=t}^T y_s, \quad (14)$$

$$\sum_{t=1}^T \sum_{k=1}^K [(z_{t,t}^k - z_{t,t-1}^k) \Delta Q_i^k(\theta^t)\theta^t + \Delta Q_i^k(\theta^t)(\theta^t - \theta^{t-1})z_{t,t-1}^k] + \sum_{t=1}^T y_t T_i(\theta^t) < 0. \quad (15)$$

Observe that

$$\sum_{t=1}^T y_t T_i(\theta^t) = \sum_{t=1}^T \sum_{s=t}^T y_s [(T(\theta^t) - T(\theta^{t-1}))]. \quad (16)$$

Now, assume that the Farkas alternative is feasible. By substituting (14) and (16) into (15),

$$\begin{aligned}
& \sum_{t=1}^T \sum_{k=1}^K [-\alpha_k \sum_{s=t}^T y_s \Delta Q_i^k(\theta^t) \theta^t + \Delta Q_i^k(\theta^t) (\theta^t - \theta^{t-1}) z_{t,t-1}^k] + \sum_{t=1}^T y_t T_i(\theta^t) \\
&= \sum_{t=1}^T \sum_{k=1}^K \Delta Q_i^k(\theta^t) (\theta^t - \theta^{t-1}) z_{t,t-1}^k - \sum_{t=1}^T (\sum_{s=t}^T y_s) \Delta Q_i(\theta^t) \theta^t + \sum_{t=1}^T (\sum_{s=t}^T y_s) (T_i(\theta^t) - T_i(\theta^{t-1})) \\
&= \sum_{t=1}^T \left[\sum_{k=1}^K \Delta Q_i^k(\theta^t) (\theta^t - \theta^{t-1}) z_{t,t-1}^k + (\sum_{s=t}^T y_s) [T_i(\theta^t) - T_i(\theta^{t-1}) - \Delta Q_i(\theta^t) \theta^t] \right], \tag{17}
\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{t=1}^T \left[\sum_{k=1}^K \Delta Q_i^k(\theta^t) (\theta^t - \theta^{t-1}) \alpha^k \sum_{s=t}^T y_s + (\sum_{s=t}^T y_s) [T_i(\theta^t) - T_i(\theta^{t-1}) - \Delta Q_i(\theta^t) \theta^t] \right], \tag{18} \\
&= \sum_{t=1}^T (\sum_{s=t}^T y_s) [T_i(\theta^t) - T_i(\theta^{t-1}) - \Delta Q_i(\theta^t) \theta^{t-1}]
\end{aligned}$$

where

$$\Delta Q_i(\theta^t) = Q_i(\theta^t) - Q_i(\theta^{t-1}).$$

The inequality between (17) and (18) is from (14). Notice that

$$\sum_{k=1}^K \Delta Q_i^k(\theta^t) (\theta^t - \theta^{t-1}) z_{t,t-1}^k \geq 0, \tag{19}$$

$$T_i(\theta^t) - T_i(\theta^{t-1}) - \Delta Q_i(\theta^t) \theta^t \leq 0, \tag{20}$$

$$T_i(\theta^t) - T_i(\theta^{t-1}) - \Delta Q_i(\theta^t) \theta^{t-1} \geq 0. \tag{21}$$

If $\sum_{s=t}^T y_s \leq 0$, then for such t ,

$$\sum_{k=1}^K \Delta Q_i^k(\theta^t) (\theta^t - \theta^{t-1}) z_{t,t-1}^k + (\sum_{s=t}^T y_s) [T_i(\theta^t) - T_i(\theta^{t-1}) - \Delta Q_i(\theta^t) \theta^t] \geq 0.$$

If $\sum_{s=t}^T y_s > 0$, then for such t ,

$$(\sum_{s=t}^T y_s) [T_i(\theta^t) - T_i(\theta^{t-1}) - \Delta Q_i(\theta^t) \theta^{t-1}] \geq 0.$$

Therefore the Farkas' alternative is infeasible and the original system is feasible. \square

Proof of Theorem 5

Proof. Denote $b_\theta = T(\theta)/Q(\theta)$ for each θ such that $Q(\theta) > 0$, where Q is the reduced form of $\{q_i\}_{i=1}^N$. We will show that $b_\theta > b_{\theta'}$ if and only if $Q(\theta) > Q(\theta')$. Consider θ^t such that $Q(\theta^t) > 0$. Since (Q, T) is incentive compatible,

$$\begin{aligned} Q(\theta^t)\theta^t - T(\theta^t) &\geq Q(\theta^{t+1})\theta^t - T(\theta^{t+1}) \\ \iff \frac{Q(\theta^t)}{Q(\theta^{t+1})} \left[\theta^t - \frac{T(\theta^t)}{Q(\theta^t)} \right] &\geq \theta^t - \frac{T(\theta^{t+1})}{Q(\theta^{t+1})} \end{aligned}$$

If $Q(\theta^t) < Q(\theta^{t+1})$, $b_{\theta^t} < b_{\theta^{t+1}}$ by the inequality. If $Q(\theta^t) = Q(\theta^{t+1})$, then $T(\theta^t) = T(\theta^{t+1})$ by the incentive compatibility. Since b_θ increases if and only if $Q(\theta)$ increases, if each bidder with type θ places a bid b_θ , it will implement the monotonic hierarchical $\{q_i\}_{i=1}^N$. The expected payment when bidding b_θ is $Q(\theta^t)b_\theta = T(\theta)$ by construction. Thus, it is an equilibrium strategy for a bidder with type θ to place a bid b_θ if $Q(\theta) > 0$. If $Q(\theta) = 0$, bidders with type θ choose to opt out in the equilibrium. \square