Efficient Bilateral Trade*

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Abstract

Can two parties reach an ex-post Pareto efficient trade agreement? The importance of the question was elucidated by Coase (1960), and Myerson and Satterthwaite (1983) provided a commonly accepted negative answer that such agreement is impossible when the parties are privately informed. We show that this negative answer depends on the assumption of quasi-linear preferences: efficient trade is possible if risk-aversion or wealth effects are sufficiently large or if agents' utility is not too responsive to private information. Under empirically-grounded specifications of risk aversion and elasticity of trade, two parties can trade efficiently despite substantial asymmetry of information.

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1 Introduction

Coase (1960) pointed out that economic agents can reach efficient agreements among themselves irrespective of the presence of externalities and without the need for government intervention so long as property rights are unambiguously enforced and there are no transaction costs. His analysis led to a rich literature on the role of transactions costs in trade.[1] A key insight of this literature is that the presence of asymmetric information creates a form of transactions costs that make reaching efficiency impossible. This insight was established by Myerson and Satterthwaite (1983) for the canonical problem of whether a seller and a buyer of an indivisible good can trade efficiently if they are privately informed about their valuations for the good and ex ante either of them might have the higher valuation.

Myerson and Satterthwaite's impossibility theorem, a central result in mechanism design, states that no Bayesian incentive-compatible, interim individually rational, non-subsidized mechanism is ex-post Pareto efficient. This impossibility theorem had a large impact on the economics literature and the practice of market design.[2] However, what is often disregarded is that this theorem assumes that agents have quasilinear utility functions. We show that, surprisingly, the impossibility of ex post Pareto efficient trade hinges on this assumption. This matters because risk aversion and wealth effects, two phenomena assumed away by quasilinearity, are important in many economic environments, particularly those involving large trades.[3]

Our results establish that ex post Pareto efficient trade among privately informed parties might be possible when the trading parties are risk averse or their utility from the object traded depends on their wealth. In Theorem 1, we show that efficient trade is generically possible as long as the asymmetry

[1] Cf. Demsetz 1968, Williamson 2010.

[2] See, for instance, Milgrom's (2004) discussion of the role the impossibility theorem played in the FCC deliberations on the first US spectrum auctions and Loertscher, Marx, and Wilkening's (2013) discussion of how the impossibility theorem led to the focus of market design on primary markets.

[3] See, for instance, Weitzman's (1977) analysis of the role of wealth effects in maximizing social welfare in the presence of wealth inequality.

of information is not too large, so that agents' utilities are not too dependent on private information. We provide a Bayesian incentive-compatible and interim individually rational mechanism that, under these conditions, is ex-post Pareto efficient and that does not rely on any subsidies or on budget breaking by third parties.[4] The restriction on private information is necessary in this general possibility result as the range of informational asymmetry that allows for efficient trade diminishes as we approach the quasilinear limit of our model.[5] In Theorem 2, we show that when risk-aversion or wealth effects are substantive, efficient trade may be possible even if agents' utilities are highly dependent on their private information. In particular, our results imply that the central impossibility insight of mechanism design hinges on the assumption of quasilinear utilities. In natural examples we show that the mechanisms we construct generate exactly efficient trade under substantial asymmetry of information when agents' risk aversion takes values observed in empirical and experimental studies of risk aversion as well as when the elasticity of substitution between the good and money takes values observed in the literature estimating such elasticities.

Why is efficient trade not possible with quasilinear preferences but might be possible when agents are risk averse? There are two parts to the explanation. First, a standard argument for the impossibility under quasilinear preferences observes that, in an incentive compatible mechanism, each agent needs to be provided with rents commensurate with her or his private information, and the gains from efficient trade are not sufficient to cover the rents of both parties. Risk aversion and wealth effects open up an additional source of efficiency gains. A seller who sold the good has more money than the seller

[4]Like Myerson and Satterthwaite we look at ex-post efficiency in the sense that we evaluate efficiency assuming that we know the agents' types. With respect to the resolution of the lotteries, our contracts are efficient not only ex post but also ex ante. We thus establish the possibility of efficient trade for traders who are already informed. Our positive result hinges on this as our Theorem 5 shows that ex-ante efficiency is in general not possible in the environments we study.

[5]We show in an example in Section 4 that near the quasilinear case, the range of asymmetric information allowing efficient trade is roughly linear in the coefficient of relative risk aversion.

who did not; if the seller's marginal utility of money is decreasing in money holdings then such a seller would prefer to have slightly more money when not selling the good and slightly less when selling the good. For the same reason, the buyer would like to pay less when receiving the good in exchange for paying something when not receiving the good. When trading parties' valuations are close to each other—the valuation range that drives Myerson-Satthertwaite's impossibility—any deterministic allocation can be Pareto improved by a lottery in which the good is transferred probabilistically and money transfers are such that each trader's money holdings (and hence marginal valuations of money) depend less on whether the trader has the good or not.[6] The role for non-degenerate lotteries does not emerge in the case of quasilinear preferences where efficiency means assigning the item to the highest-value agent with probability one.

Second, the lotteries present in efficient trade mechanisms allow us to balance agents' incentives in ways that are absent in the quasilinear setting. In the quasilinear setting of Myerson and Satterthwaite, truth-telling cannot be incentive compatible. If the seller overstates his value by a little bit, it results in a higher price when he sells, a first-order gain, but the accompanying small reduction in the probability of sale only arises in the case where he is almost indifferent to selling, a second order loss. In our setting, the reported type affects the probability of trade, even when the seller is far from indifferent between trading or not. In effect, the gains and losses from misreporting are both of the first order creating countervailing incentives for the trading parties.[7]

To establish our possibility results, we need to develop a methodology to study Pareto efficient trading mechanisms in settings with risk-averse agents. For instance, in symmetric settings, the above discussion shows that Pareto efficiency requires randomization when agents' valuations for the good are nearly equal; thus the mechanism design problem we study is very different from the

[6]Randomization is also shown to be welfare improving in other settings with non-quasilinear preferences e.g. in competitive equilibrium analysis of markets with indivisible labor (Rogerson, 1988) and optimal education financing (Garratt and Marshall, 1994).

[7]We thank James Peck for suggesting the explanation in terms of countervailing incentives.

one studied by Myerson and Satterthwaite precisely in the range of types that underlies their impossibility result. Unlike in their quasilinear setting, in ours, the size of money transfers (which are conditional on the allocation of the good) are uniquely determined by efficiency, but the allocation of the good itself is not. The key to constructing the Pareto efficient mechanism is a judicious choice of the probability of allocating the good to each of the trading agents; in a direct mechanism the probability needs to respond to agents' reports in a way that ensures that truthful reporting is Bayesian incentive compatible and interim individually rational. We reduce the problem of constructing such a probability function to a non-standard system of partial differential equations and offer a constructive way to solve this system of equations.

While we focused so far on risk averse agents, in the paper we also demonstrate that the possibility of efficient trade does not require risk aversion if there is synergy between having the good and having money, for instance, when agents' utility from the good and money take Cobb-Douglas form. Sufficient synergy allows the agents to trade efficiently regardless of the extent of informational asymmetry (Theorem 3). Furthermore a simple lottery mechanism implements than efficient trade in dominant strategies. This is possible when synergy implies that a winner-take-all allocation is efficient; we show that efficiency can then be achieved without the need to elicit types. In settings where an interior solution is required for efficiency, both dominant-strategy and ex-post implementation are generically impossible (Theorem 4).

The question as to when efficient trade is possible is important and has been extensively studied. Myerson and Satterthwaite show that efficiency can be obtained in their quasi-linear setting provided the distribution of types is continuous and the buyer's value is always higher than the seller's value. Williams (1999) extend the impossibility result to symmetric settings with many agents while Makowski and Mezzetti (1994) show that for an open set of asymmetric distributions trade can be efficient provided there are multiple buyers. Gresik and Satthertwaite (1983), Wilson (1985), Makowski and Ostroy (1989), Rustichini, Satterthwaite, and Williams (1994), Reny and Perry (2006), Cripps and Swinkels (2006), Jantschgi et al (2022),and many others establish that trade is asymptotically efficient as the number of buyers and sellers becomes large (this literature studies both interim and ex post efficiency). McAfee (1991) shows that efficient trade can be possible when the ex ante gains from trade are large and the two trading parties have access to an uniformed budget breaking third agent.[8]

At the same time, we know of no successful attempt to go beyond Myerson and Satthertwaite (1983) and demonstrate the possibility of fully efficient mechanisms in their original context of two expected-utility maximizing agents, one buyer and one seller.[9] What was demonstrated is the possibility of approximate efficiency in two contexts. Chatterjee and Samuelson (1983) show that double-auctions are asymptotically efficient in the limit as the agents become infinitely risk-averse; in contrast we establish efficiency for empirically relevant levels of risk aversion.[10] And, McAfee and Reny (1992) show that when private values are correlated (and a hazard rate assumption is satisfied) a judicious use of Cremer and McLean (1988) lotteries allows the parties to reduce their incentives to misreport so as to permit outcomes as close to efficiency as desired; in contrast, our results allow for both independence and correlation, and they do not rely on large bets in the spirit of Cremer and McLean (all lotteries we employ are bounded by agents' wealth levels).

While ours is likely the first paper to study efficient Bayesian incentive compatible bilateral trade mechanisms allowing for risk aversion and wealth effects,

[8] McAfee (1991) studied the problem of trading divisible goods; see (Riley 2012) for an analysis with indivisible goods.

[9] Cramton, Gibbons, and Klemperer (1987) show that the buyer-seller ownership structure is essential to the impossibility result. When both trading parties initially own shares of the the traded good, full efficiency might be possible already within the quasilinear framework.

[10] Chatterjee and Samuelson (1983) also show that double auction is not efficient, and Chatterjee (1982) shows that no mechanism in a large class of incentive compatible mechanisms is efficient; these two papers are important precursors of Myerson and Satterthwaite's impossibility theorem. After Myerson and Satthertwaite, many authors provided alternative proofs of their impossibility result, including Williams (1999) and Krishna and Perry (1998). Matsuo (1989) studied discrete distributions, while others, e.g., Copic and Ponsati (2016) and Blumrosen and Dobzinski (2021), studied the efficiency loss of simple mechanisms. Simultaneous or following ours work looks at bilateral trade among behavioral agents: Wolitzky (2016) studies max-min agents, Crawford (2021) k-level agents, and Benkert (2022) loss-averse agents.

it follows a rich mechanism design literature studying risk aversion. This literature generally assumes that the seller is uninformed and it often restricts attention to deterministic mechanisms (e.g. Holt (1980) and Morimoto and Serizawa (2014)), or focuses on revenue-maximization (e.g. Matthews (1983), Maskin and Riley (1984)).[11] The closest prior paper is Garratt (1999) who studies the Pareto frontier and shows that random mechanisms can dominate deterministic ones in a complete information setting.

The model and assumptions are presented in Section 2. The possibility results are presented in Sections 3 and 4. These results are complemented by additional results in Section 5 that cover ex-post and strategy-proof implementation and ex-ante efficiency, and three examples in Section 6. The first two examples in Section 6 show how to use the methodology we developed to study two canonical utility specifications: constant relative risk aversion (CRRA) and constant absolute risk aversion (CARA). The third example shows that with Cobb-Douglas preferences efficient trade is possible for any distribution of agents' types, and for any distribution of agents' initial wealth levels. In this final example, efficient trade is ex post implementable and the mechanism works regardless of whether or not endowed money holdings are private information. Section 7 contains our concluding remarks.

2 Model and Assumptions

A seller is endowed with an indivisible good and money endowment m_s while the buyer has money endowment m_b.[12] Aggregate money holdings are denoted by $M = m_s + m_b$. We assume that $M > 0$ and an agent's money holdings before and after trade belong to $[0, M]$. The seller privately knows their type (cost) $c \in [\underline{c}, \overline{c}] \subset \mathbb{R}_+$ and the buyer privately knows their type (value) $v \in [\underline{v}, \overline{v}] \subset \mathbb{R}_+$. The types are drawn from a joint distribution on $[\underline{c}, \overline{c}] \times [\underline{v}, \overline{v}]$;

[11] More recent work in this line includes Kazumura, Mishra, Serizawa (2020).

[12] We formulate our results in terms of trades in an indivisible good and money, thus upholding all elements of the Myerson and Satthertwaite's environment except for the traders' utility functions, but we could, of course, replace money with a divisible good or a basket of goods.

this distribution is commonly known and otherwise arbitrary. We assume the utility function $u(x, m, \theta)$ of each agent is twice continuously differentiable in money (m) and in type (θ), strictly increasing in having the indivisible good (x), weakly increasing in type when $x = 0$ and strictly increasing in type when $x = 1$, strictly increasing and strictly concave in money, and that it satisfies the Inada condition, $\frac{\partial}{\partial m} u(x, 0, \theta) = +\infty$ for $x \in \{0, 1\}$ and all types θ.[13] For notational convenience, we extend the utility function to lotteries over the good: $u(x, m, \theta) = xu(1, m, \theta) + (1 - x) u(0, m, \theta)$ for $x \in [0, 1]$.

For simplicity of exposition, we assume that the good is normal in the sense of Cook and Graham 1977: for any type θ and any relevant money levels $m, p, \epsilon > 0$, if $u(0, m, \theta) = u(1, m - p, \theta)$ then $u(0, m + \epsilon - p, \theta) < u(1, m + \epsilon, \theta)$. Normality is an empty condition if there is no price p at which $u(0, m, \theta) = u(1, m - p, \theta)$; if there is such a price, then normality implies that it is unique.[14] The normality assumption for an indivisible good is the natural counterpart of normality for divisible goods, as it means that the more money an agent has, the more they are willing to pay for the good. As normality requires that each player's reservation price for the good strictly increases with the agent's wealth, it does not hold under quasilinear utility.

We also impose the following assumption on how agents' utilities respond to their types: there exists a function $\psi(x, \theta)$ such that for any $x \in [0, 1]$, $m \in [0, M]$, and any type θ, we have

$$\frac{\partial}{\partial \theta} \log \left(\frac{\partial}{\partial x} u(x, m, \theta) \right) > \frac{\partial}{\partial \theta} \log \left(\frac{\partial}{\partial m} u(x, m, \theta) \right) = \psi(x, \theta). \tag{1}$$

That is, we want the type-elasticity of the marginal value of the good to exceed

[13] The differentiability, strict concavity and Inada conditions simplify the analysis, but as we show in Theorem 3, these conditions can be relaxed if there is sufficient synergy between the good and money.

[14] The normality assumption allows us to focus on a single region of nonconcavity of the Pareto frontier, as we discuss below. It is not required, and can be relaxed by assuming that there is a finite partition of wealth levels such that each player's reservation price for the good is strictly monotonic on each element of the partition. Relaxing normality in this way would require us to modify Theorem 1 and Corollaries 1 and 2 and exclude a finite set of money profiles, rather than just one.

the type-elasticity of the marginal utility of money and the latter elasticity to be constant in money. Both components of this assumption are automatically satisfied when utilities are quasilinear in money, and higher types have higher utility from consuming the good. The assumption is also satisfied when agents' utilities are additively separable, i.e., $u(x, m, \theta) = \theta x + v(m)$. Furthermore, this assumption is only needed for our analysis of the second-order condition of the mechanism we construct, where it is sufficient—and guarantees that the second-order condition is satisfied irrespective of value distributions—but is is not necessary and in the presence of distributional assumptions can be substantially relaxed.

Our assumptions on utility determine some features of the Pareto frontier (see Figure 1 noting that the origin in this and other similar figures is not necessarily the point (0,0); it is the point $(u(1,0,v), u(1,0,c))$.[15] To describe

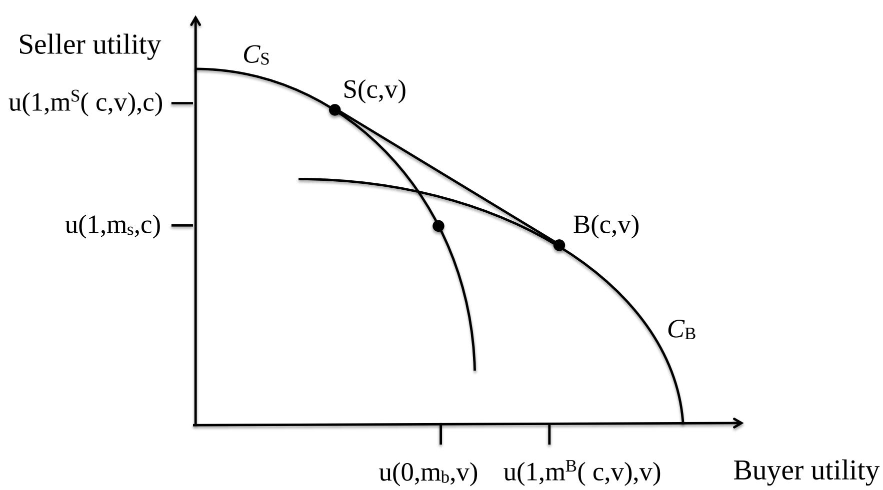

Figure 1: Pareto Frontier with a Normal Good.

the frontier, let us fix agents' types c and v. The frontier is the upper envelope of the curves

$$\mathcal{C}_S = \{(u(1, m, c), u(0, M - m, v)) : m \in [0, M]\}$$

[15] See Garratt (1999) for a more detailed discussion of the Pareto frontier for normal goods.

and

$$\mathcal{C}_B = \{(u(0, m, c), u(1, M - m, v)) : m \in [0, M]\}.$$

The curve $\mathcal{C}_S$ traces the utilities when the seller has the good, while the curve $\mathcal{C}_B$ traces the utilities when the buyer has the good. Since we assume that it is better to have the good than not to have it, the curve $\mathcal{C}_S$ starts higher than $\mathcal{C}_B$ on the axis of seller's utilities (the vertical axis), and $\mathcal{C}_S$ ends lower than $\mathcal{C}_B$ on the axis of buyer's utilities (the horizontal axis). Thus, as we move along the Pareto frontier from the seller's most preferred point to the buyer's most preferred point, we start on the curve $\mathcal{C}_S$ and we end on the curve $\mathcal{C}_B$. The two curves might but do not need to intersect. If the curves $\mathcal{C}_S$ and $\mathcal{C}_B$ intersect, as drawn, then normality implies that $\mathcal{C}_S$ intersects $\mathcal{C}_B$ only once and that it does so from above. Further, the point at which they intersect cannot be part of the frontier because normality implies that at this point $\mathcal{C}_S$ intersects $\mathcal{C}_B$ strictly from above and hence a randomization over any point just to the left of the intersection and any point just to the right of the intersection is strictly preferred to the intersection point by both trading parties. The frontier thus contains a flat part consisting of randomizations between two points: $S(c, v) \in \mathcal{C}_S$ and $B(c, v) \in \mathcal{C}_B$, where the seller strictly prefers $S(c, v)$ to $B(c, v)$ and the buyer strictly prefers $B(c, v)$ to $S(c, v)$. The strict concavity of u in money implies that these two points are uniquely determined. We call them the critical points. If the curves do not intersect, then the frontier—being an upper envelope—also contains a flat part consisting of randomizations between two points: $S(c, v) \in \mathcal{C}_S$ and $B(c, v) \in \mathcal{C}_B$, where the seller strictly prefers $S(c, v)$ to $B(c, v)$ and the buyer strictly prefers $B(c, v)$ to $S(c, v)$. In both cases, strict concavity of the utility function and the Inada condition ensure that the flat part is tangent to the curves $\mathcal{C}_S$ and $\mathcal{C}_B$, as shown in Figure 1, which illustrates the former case.

Let $m^S(c, v)$ denote the seller's money holdings at $S(c, v)$ and $m^B(c, v)$ denote the buyer's money holdings at $B(c, v)$. The critical levels of money holdings depend upon agents' reports of their types. By the Inada condition, the critical points $S(c, v)$ and $B(c, v)$ are in $(0, M)$, and the money levels

$m^S(c,v)$ and $m^B(c,v)$ are uniquely determined by the following equations

$$\frac{\frac{\partial}{\partial m}u\left(1,m^S,c\right)}{\frac{\partial}{\partial m}u\left(0,M-m^S,v\right)}=\frac{\frac{\partial}{\partial m}u\left(0,M-m^B,c\right)}{\frac{\partial}{\partial m}u\left(1,m^B,v\right)}=\frac{u\left(1,m^S,c\right)-u\left(0,M-m^B,c\right)}{u\left(1,m^B,v\right)-u\left(0,M-m^S,v\right)}. \tag{2}$$

These equations express the fact that the randomization interval is tangent to the Pareto frontier at both critical points $S(c,v)$ and $B(c,v)$. The dependence of the money holdings $m^S(c,v)$ and $m^B(c,v)$, on (c,v) is continuously differentiable by the Implicit Function Theorem and the assumptions we imposed on u. In Appendix B, we show that our assumptions also imply that

$$\frac{\partial}{\partial c}m^B(c,v)>0>\frac{\partial}{\partial c}m^S(c,v) \text{ and } \frac{\partial}{\partial v}m^S(c,v)>0>\frac{\partial}{\partial v}m^B(c,v). \tag{3}$$

Thus, each agent's money holding at their preferred critical point of the Pareto frontier is decreasing in their own type and increasing in the other agent's type.

3 The First Possibility Theorem on Efficient Trade

We now show that efficient trade is possible for the class of utility functions described in Section 2 if the asymmetry of information is sufficiently bounded. For an agent with type space Θ, we say that the asymmetry of information is bounded by $\Delta\geq 0$ if there exists a type $\theta^*\in\Theta$ such that Δ is weakly larger than the maximum of $|u(x,m,\theta)-u(x,m,\theta^*)|$ over all $\theta\in\Theta$, $m\in[0,M]$, and $x\in\{0,1\}$.

Theorem 1. *For any seller's type and buyer's type, any utility functions of these types, any sum of money endowments $M>0$, and any profile of money endowments but one, there is $\Delta>0$ such that if the seller's and buyer's asymmetry of information is bounded by Δ, then there is an incentive-compatible, individually-rational, and budget-balanced mechanism that generates efficient trade.*

As an immediate corollary we have

Corollary 1. *Fix a type profile* $(c^*, v^*) \in (0,\infty)^2$*, the sum of money endowments* $M > 0$*, and the utility function* u*. For any profile of nonnegative money endowments but one, there are intervals* $(\underline{c}, \overline{c}) \ni c^*$ *and* $(\underline{v}, \overline{v}) \ni v^*$ *such that: if agents draw their types from arbitrary distributions on* $(\underline{c}, \overline{c}) \times (\underline{v}, \overline{v})$*, then there is an incentive-compatible, individually-rational, and budget-balanced mechanism that generates efficient trade.*

This corollary follows immediately from Theorem 1 because, for any $\Delta > 0$, by taking the intervals $(\underline{c}, \overline{c}) \ni c^*$ and $(\underline{v}, \overline{v}) \ni v^*$ to be sufficiently small we can ensure that the asymmetry of information is bounded by Δ. A special case of interest obtains when $c^* = v^*$; in that case the corollary establishes efficient trade for any distribution of agents' types on $\left(\underline{\theta}, \overline{\theta}\right)^2$ for some interval $\left(\underline{\theta}, \overline{\theta}\right) \ni \theta^*$, where θ^* is the common value of c^* and v^*.

For a given utility function u and aggregate money holdings M, the range of types that permit efficient trade can be quite large. This aspect of the result is illustrated by the examples of Section 6. In Example 1 (Log utility) efficient trade is achieved for types drawn uniformly from $[2, 100]$, in Example 2 (CARA utility) efficient trade is achieved for types drawn from $[1, 2]^2$, and in Example 3 (Cobb-Douglas utility) efficient trade is achievable for any distribution of types in $[0,\infty)^2$. In general, the range in which efficient trade is possible depends on the distance from quasilinearity. We examine this dependence in the next section and show that efficient trade is possible on large type domains for empirically grounded specifications of trading parties' risk aversion or the elasticity of substitution between money and the good.

Having already discussed the intuition behind the possibility of efficient trade in the Introduction, we now turn to the main steps of the proof, with the more technical parts relegated to the Appendix.

3.1 Proof of Theorem 1

Let us arbitrarily fix the seller's type $c^* \in [\underline{c}, \overline{c}]$ and buyer's type $v^* \in [\underline{v}, \overline{v}]$, the utility functions of these types, and a distribution of endowed money holdings in which the seller's initial money holding is anything but $m^S(c^*, v^*)$.[16] Three cases are possible depending on how much money the seller initially has.

The no-trade case. Suppose the seller's endowed money holding is strictly greater than $m^S(c^*, v^*)$. Then, it must also be true that for utilities close to $u(1, m_s, c^*)$ for the seller and $u(0, m_b, v^*)$ for the buyer that the no-trade mechanism is efficient. Hence there is $\Delta > 0$ that guarantees that the no-trade mechanism is efficient and satisfies all our other requirements.

The sure-trade case. Suppose the seller's endowment is such that the seller's initial utility level is strictly below $u\left(0, M - m^B\left(c^*, v^*\right), c^*\right)$. Then there is a point on the Pareto frontier of (c^*, v^*) that strictly dominates the agents' utility at the initial endowments (point F in Figure 2). At this point on the frontier the seller has no good and has money holdings $m^S + t$ for some constant transfer t, while the buyer has the good and money holdings $m^B - t$. In particular, the pre-trade seller's utility is strictly below $u\left(0, m^S + t, c^*\right)$ and the pre-trade buyer's utility is strictly below $u\left(1, m^B - t, v^*\right)$. These bounds on agents' pre-trade utility remain true for type profiles close to (c^*, v^*). There is then $\Delta > 0$ that guarantees that the mechanism that allocates the good and money $m^B - t$ to the buyer and money $m^S + t$ (without good) to the seller is Pareto efficient, individually rational, and does not require a subsidy. Furthermore, this mechanism is incentive compatible as it does not rely on agents reports.[17]

The type-dependent-trade case. Suppose the seller's endowment is intermediate: their initial money holding is strictly below $m^S\left(c^*, v^*\right)$, and their initial utility level is weakly above $u\left(0, M - m^B\left(c^*, v^*\right), c^*\right)$. The efficient trade direct mechanism we construct takes the following form: with probability $\pi\left(c, v\right)$, the seller has the good and money holdings $m^S\left(c, v\right)$, while the buyer

[16]It follows that the buyer's initial money holding is anything but $M - m^S(c^*, v^*)$. While types c^* and v^* need not be the same as the reference types in the Δ-bound on asymmetric information, it is without loss of generality to assume that they are these reference types, because if asymmetric information is bound by Δ relative to some type then it is bound by 2Δ relative to any other type.

[17]A similar fixed-terms-of-trade mechanism delivers efficient trade whenever there exists a point on the Pareto frontier that is strictly preferred by both the buyer and seller to status quo, and strictly more favorable to the buyer than having the good and money $m^B\left(c^*, v^*\right)$.

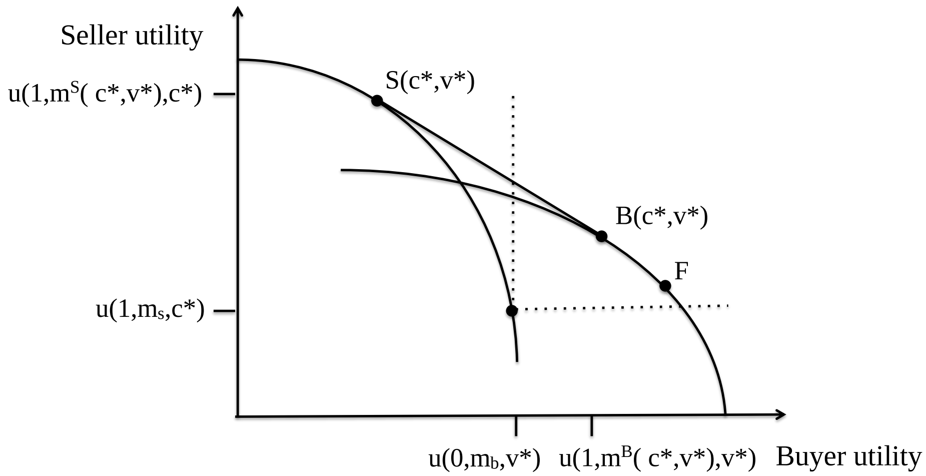

Figure 2: Individually Rational Part of the Pareto Frontier. The Case of Trade at Fixed Price.

has the remaining money $M - m^S(c,v)$; with probability $1-\pi(c,v)$, the buyer has the good and money holdings $m^B(c,v)$, while the seller has the remaining money $M - m^B(c,v)$. We set $\pi(c^*,v^*) = \pi^* \in (0,1)$ so that the pair of agents' expected utilities give us a point $F = F(c^*,v^*;\pi^*)$ on the flat part of the frontier, strictly between $S(c^*,v^*)$ and $B(c^*,v^*)$ (see Figure 3), that is strictly preferred by the buyer and the seller to the initial situation, $(u_s(1,m_s,c^*), u_b(0,m_b,v^*))$. The assumptions of the present case ensure that such a π^* exists. By making $\Delta > 0$ sufficiently small, we can ensure that for all type profiles (c,v) the critical money holdings $m^S(c,v)$ and $m^B(c,v)$ are sufficiently close to $m^S(c^*,v^*)$ and $m^B(c^*,v^*)$, respectively, so that the point $F(c,v;\pi(c,v))$ on the frontier selected by our mechanism is strictly preferred by the agents to the initial situation when π is sufficiently close to π^*.

The crux of the reminder of the argument is to construct a function π whose values are close to π^* when Δ is small, and that makes the above mechanism Bayesian incentive compatible. Incentive compatibility means that for the seller

$$\Pi^S(c,\hat{c}) = E_v\left(\pi(\hat{c},v)\, u\left(1, m^S(\hat{c},v), c\right) + (1-\pi(\hat{c},v))\, u\left(0, M - m^B(\hat{c},v), c\right)\right)$$

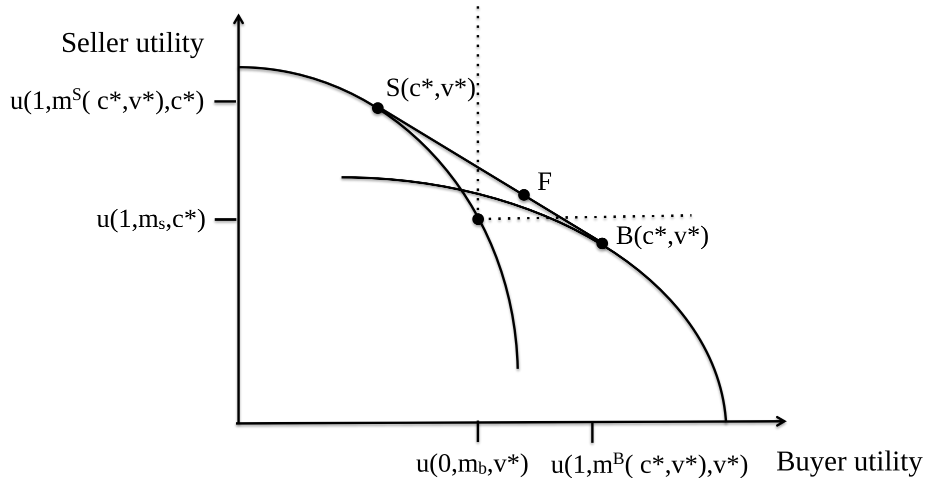

Figure 3: Individually Rational Part of the Pareto Frontier. The Case of Trade at Varying Prices.

is maximized at $\hat{c} = c$, and similarly, for the buyer,

$$\Pi^B(v, \hat{v}) = E_c\left(\pi(c, \hat{v})\, u\left(0, M - m^S(c, \hat{v}), v\right) + (1 - \pi(c, \hat{v}))\, u\left(1, m^B(c, \hat{v}), v\right)\right)$$

is maximized at $\hat{v} = v$. We guarantee these properties by constructing π so that truthful reporting satisfies the agents' first-order conditions. Solving the first-order conditions is sufficient for incentive compatibility because—as we show in Appendix C.1—condition (1) implies that the second-order condition of each agent's maximization problem is satisfied at every point at which the first-order condition is satisfied.

Assuming truthful reporting by the other agent, the first-order condition for the seller is

$$0 = E_v\left\{\frac{\partial}{\partial \hat{c}}\pi(\hat{c}, v)\, u\left(1, m^S(\hat{c}, v), c\right) + \pi(\hat{c}, v)\frac{\partial}{\partial m}u\left(1, m^S(\hat{c}, v), c\right)\frac{\partial}{\partial \hat{c}}m^S(\hat{c}, v)\right.$$
$$\left. -\frac{\partial}{\partial \hat{c}}\pi(\hat{c}, v)\, u\left(0, M - m^B(\hat{c}, v), c\right) - (1 - \pi(\hat{c}, v))\frac{\partial}{\partial m}u\left(0, M - m^B(\hat{c}, v), c\right)\frac{\partial}{\partial \hat{c}}m^B(\hat{c}, v)\right\} \quad (4)$$

and the first-order condition for the buyer is

$$0 = E_c\Bigg\{\frac{\partial}{\partial \hat{v}}\pi(c,\hat{v})\,u\left(0, M - m^S(c,\hat{v}), v\right) - \pi(c,\hat{v})\frac{\partial}{\partial m}u\left(0, M - m^S(c,\hat{v}), v\right)\frac{\partial}{\partial \hat{v}}m^S(c,\hat{v})$$
$$-\frac{\partial}{\partial \hat{v}}\pi(c,\hat{v})\,u\left(1, m^B(c,\hat{v}), v\right) + (1-\pi(c,\hat{v}))\frac{\partial}{\partial m}u\left(1, m^B(c,\hat{v}), v\right)\frac{\partial}{\partial \hat{v}}m^B(c,\hat{v})\Bigg\}. \tag{5}$$

We want $\hat{c} = c$ to satisfy the seller's first-order condition and $\hat{v} = v$ to satisfy the buyer's first-order condition. Hence, the two conditions give us a system of partial differential equations (PDEs) on $\pi(c,v)$ of the form analyzed in the following lemma.

Lemma 1. *Let I be a bounded interval of positive length and let F be a joint distribution of (c,v) over domain $[\underline{c},\overline{c}] \times [\underline{v},\overline{v}] \subseteq \mathbb{R}^2$. Let $S_1(\cdot,\cdot), S_2(\cdot,\cdot)$ and $B_1(\cdot,\cdot), B_2(\cdot,\cdot)$ be functions defined on $[\underline{c},\overline{c}] \times [\underline{v},\overline{v}]$, and ϕ be a function on $[\underline{c},\overline{c}]$ and ψ on $[\underline{v},\overline{v}]$. Suppose that all these functions are continuously differentiable and that $S_1, B_1 \neq 0$ for all arguments (c,v). Then, for any $(c^*,v^*) \in [\underline{c},\overline{c}] \times [\underline{v},\overline{v}]$, $\pi^* > 0$ and boundary condition $\pi(c^*,v^*) = \pi^*$, the system of PDEs*

$$E_v\left[S_1(c,v)\frac{\partial}{\partial c}\pi(c,v) + S_2(c,v)\pi(c,v)\right] = \phi(c), \tag{6}$$

$$E_c\left[B_1(c,v)\frac{\partial}{\partial v}\pi(c,v) + B_2(c,v)\pi(c,v)\right] = \psi(v) \tag{7}$$

has a solution π.

The lemma is of independent interest. It tells us that for any marginal distributions of the linear PDE formulas from the lemma, we can find a function that implements these marginal distributions.[18]

[18] We have not been able to find this lemma in the literature on partial differential equations. The sufficient conditions for existence of solutions of non-averaged linear PDEs of Thomas (1934) and Mardare (2007) can easily tell us that the lemma is true if $\frac{\partial}{\partial v}\frac{S_2}{S_1} = \frac{\partial}{\partial c}\frac{B_1}{B_2}$, which is satisfied for instance when the coefficients B_i, S_i are all constant, but they are not satisfied in the general case we consider here (which is not surprising as it is much easier to satisfy the PDEs on average than it is to satisfy them pointwise).

We prove this lemma in Appendix C.4 establishing along the way a constructive procedure for finding the PDEs solutions.[19] The lemma's assumptions are satisfied by the first-order conditions above, as we verify in Appendix C.2. Furthermore, in Appendices C.3 and C.5 we show that that by controlling Δ we can guarantee that π takes values close to π^*. We can thus ensure that $\pi(c,v) \in [0,1]$ and that individual rationality is satisfied. This concludes the proof.

4 The Second Possibility Theorem on Efficient Trade: Risk Aversion and the Distance from Quasilinearity

As the results from the previous section indicate, any departure from quasilinear preferences results in efficient trade for sufficiently constrained environments. Corollary 1 expresses these constraints in terms of the range of player types. An alternative approach is to fix the non-preference primitives and ask whether a sufficient departure from quasilinearity is enough to ensure efficient trade. We demonstrate this possibility for the case where agents have private information about their valuation for the good, but their marginal utility of money is commonly known. More specifically, we assume the agents' utility takes the following separable form

$$u(x,m,\theta) = \theta x + U(m),$$

where U is strictly increasing, concave, and twice differentiable.[20]

Theorem 2. *Fix any continuous joint distribution of types on a compact type space in $(0,\infty)^2$ and any non-equal and nonnegative money endowments m_s*

[19]Examples 1 and 2 in Section 6 illustrate this procedure.

[20]For simplicity we focus on the case where U and the type space is common to both agents. This restriction is not crucial and an analogous theorem (in which the inequality of budgets is replaced by another excluded budget configuration) can be proven via an analogous proof.

and m_b. There is $\Delta > 0$ such that for all increasing, concave, and twice differentiable functions U such that the coefficient of absolute risk aversion $|U''(m)|/U'(m) \geq \Delta$ for almost every $m \in (0, m_s + m_b)$, agents with utilities $u(x, m; \theta) = \theta x + U(m)$ can trade efficiently.

In the case of CARA utility $U(m) = \frac{1-e^{-\alpha m}}{\alpha}$, Theorem 2 says that efficient trade is possible for a sufficiently large degree of risk aversion (i.e., sufficiently large α). In the limit, as α goes to 0, CARA utility becomes linear. The larger α is, the farther the utility is from quasilinearity. Hence, the result says that efficient trade is guaranteed if each agent's utility is sufficiently far away from quasilinearity. Example 2 in Section 6 illustrates this result by fixing the money endowments and type distributions and showing that $\alpha \geq .2$ is sufficient to ensure efficient trade.[21]

We prove Theorem 2 in the Appendix. The argument builds on a natural parallelism—via change of units of measurement—between imposing a high Arrow-Pratt measure of absolute risk aversion and having a small interval of types. This parallelism allows us to prove Theorem 2 using the methodology we developed for Theorem 1. In cases where the initial money holding of the seller exceeds that of the buyer, the problem trivializes as no trade is efficient for high levels of risk aversion (high Δ). In the complementary case, where the initial money holding of the buyer exceeds that of the seller, the analysis is subtler. For instance, sure trade is not adequate even in the risk-aversion limit e.g. if there is not enough money in the system to compensate the seller for parting with the good. It is also not sufficient to sell a fixed probability of the good at a fixed price as such mechanisms are in general not going to be efficient because for probability of trade in $(0, 1)$, efficiency requires the money holdings to be responsive to the type profile.

Because m is bounded above by total money holdings M, Theorem 2 has the following immediate corollary for relative risk aversion coefficients.

[21] In their seminal experimental analysis of risk aversion, Holt and Laury (2002) find that $\alpha = .2$ is the coefficient of absolute risk aversion that best describes their subjects. The coefficient of absolute risk aversion is however not invariant to changes in the unit of accounting and in general the literature finds the CRRA specification to better describe attitudes to risk; we discuss CRRA traders below.

Corollary 2. *Fix any continuous joint distribution of types on a compact type space in* $(0,\infty)^2$ *and any non-equal and nonnegative money endowments* m_s *and* m_b*. There is* $\Delta > 0$ *such that for all increasing, concave, and twice differentiable functions* U *such that the coefficient of relative risk aversion* $|mU''(m)|/U'(m) \geq \Delta$ *for almost every* $m \in (0, M)$*, agents with utilities* $u(x,m;\theta) = \theta x + U(m)$ *can trade efficiently.*

To see how the possibility of efficient trade depends on the relative risk aversion and on the elasticity of substitution between good and money, consider trade between a seller and a buyer whose utility is separable in good and money and whose utility of money exhibits CRRA: $u(x,m,\theta) = \theta x + \frac{m^{1-\gamma}}{1-\gamma}$. Given the indivisibility of the good traded, $x \in \{0,1\}$, we can interpret the CRRA utility specification as constant elasticity of substitution $u(x,m,\theta) = \theta x^{1-\frac{1}{\rho}} + \frac{1}{1-\gamma} m^{1-\frac{1}{\rho}}$ where $\rho = \frac{1}{\gamma}$ is the elasticity of substitution.

The analysis we developed in the proof of Theorem 1 allows us to conclude that when the buyer holds all the money, and prior to trade the utility of the seller and buyer are equal for types $c^* = v^*$, then, for sufficiently small range $[\underline{c}, \overline{c}] \times [\underline{v}, \overline{v}] \ni (c^*, v^*)$ over which cost and value types c, v are uniformly distributed, the following mechanism is Bayesian incentive-compatible, interim individually rational, and its outcome is always ex post Pareto efficient.[22] Given total money holdings M, this mechanism generates money holdings

$$m^B(c,v) = \frac{c^{\frac{1}{\gamma}}}{v^{\frac{1}{\gamma}} + c^{\frac{1}{\gamma}}} M \quad \text{and} \quad m^S(c,v) = \frac{v^{\frac{1}{\gamma}}}{v^{\frac{1}{\gamma}} + c^{\frac{1}{\gamma}}} M,$$

and assigns the good with probability

$$\pi(c,v) = \frac{1}{2} + \int_{c^*}^{c} \frac{1}{\gamma x} M^{1-\gamma} \int_{\underline{v}}^{\overline{v}} \frac{v^{\frac{1}{\gamma}-1} x^{\frac{1}{\gamma}-1}}{\left(v^{\frac{1}{\gamma}} + x^{\frac{1}{\gamma}}\right)^{2-\gamma}} \frac{1}{\overline{v} - \underline{v}} dv dx - \int_{v^*}^{v} \frac{1}{\gamma x} M^{1-\gamma} \int_{\underline{c}}^{\overline{c}} \frac{x^{\frac{1}{\gamma}-1} c^{\frac{1}{\gamma}-1}}{\left(x^{\frac{1}{\gamma}} + c^{\frac{1}{\gamma}}\right)^{2-\gamma}} \frac{1}{\overline{c} - \underline{c}} dc dx.$$

[22] Our construction provides an efficient trading mechanism; this mechanism is not unique. For brevity we omit the analysis establishing the properties of the mechanism presented as this analysis follows the exact same steps as the detailed analysis we provide in Examples 1 and 2 of Section 6. Neither the symmetry embedded in $c^* = v^*$ nor uniform distributions play substantive role in the analysis. The assumptions on money and utility can be replaced with requiring that in a neighborhood of $c^* = v^*$ the efficient and individually-rational assignment of the good and money depends on parties' private information.

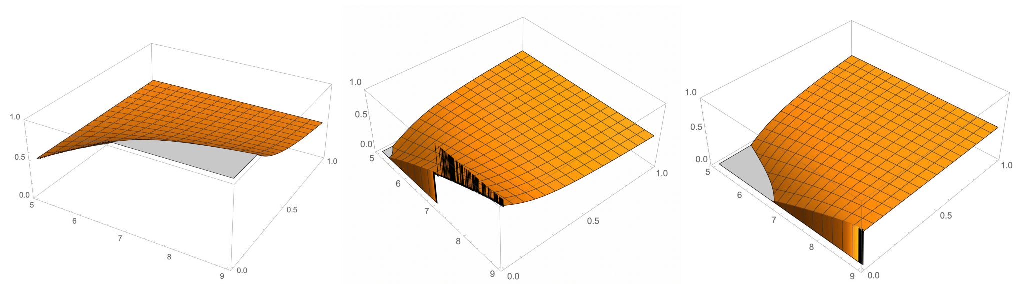

Figure 4: The probability (vertical axis) of the seller keeping the good as a function of $c \in [5,9]$ and $\gamma \in (0,1)$ for $v = 5$ (left), $v = 7$ (center), and $v = 9$ (right).

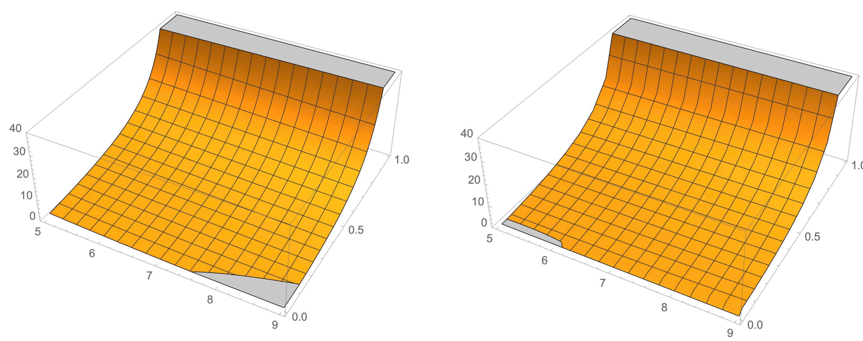

Figure 5: The expected gain (vertical axis) of the seller from participating in the mechanism as the function of $c \in [5,9]$ and $\gamma \in (0,1)$ (left); the expected gain of the buyer from participating in the mechanism as the function of $v \in [5,9]$ and $\gamma \in (0,1)$ (right).

The dependence of assignment probability π on c, v and γ is depicted in Figure 4. The figure shows that π stays between 0 and 1 for $\gamma \geq .226$; thus the mechanism is well defined for all these values of relative risk aversion.[23] Figure 5 shows that, for the same values of γ, the mechanism also raises the utility of both buyer and seller, and hence participation is individually rational for both trading parties.

Efficient trade is possible in the environment studied as long as the co-

[23] Note that π is increasing in c and decreasing in v and hence to conclude that $\pi \in [0,1]$ it is sufficient to check check $\pi(5,9)$ (illustrated on the left sub-figure) and $\pi(9,5)$ (illustrated on the right subfigure). This and subsequent figures for CRRA utilities are drawn in Mathematica.

efficient of relative risk aversion $\gamma \geq .226$, or, equivalently, as long as the coefficient of the elasticity of trade $\rho \leq 4.425$. These are empirically relevant values of risk aversion and elasticity. For instance, Campbell (2018) reports that rationalizing the equity premium puzzle leads to the values of relative risk aversion between 5 and 36 in conservative estimates (with less conservative estimates requiring even higher γ); he also reports similarly high values of γ as required to rationalize the risk-free rate puzzle. Other analyses of the field data generate lower—but still higher than .226—estimates of γ.[24] Estimates of γ higher than .226 estimates are also obtained in experimental investigations of risk aversion. Holt and Laury (2002) estimate γ as being between .3 and .5 in low-stake experiments and .7 and 1 in high-stake experiments; these estimates are in line with other experiments and field data they review. Also studies of trade elasticity give us values at which efficient trade in the above example is possible. A thorough evaluation of the elasticity of substitution by Simonovska and Waugh (2014) puts the benchmark elasticity at $\rho = 4.14$; they report the range of estimates across a variety of datasets as 2.79 to 4.46.

The CRRA example allows us to see what happens as the utility function approximates the quasilinear case ($\gamma \to 0$), thus complementing our discussion of the quasilinear setting of Myerson and Satterthwaite that we presented in the Introduction. The key constraint on the ability to trade efficiently in the CRRA example as $\gamma \to 0$ is the need to shift the probability of having the good around the point $v = c$; the speed with which this happens explodes to infinity as $\gamma \to 0$, thus making it impossible to keep the probability $\pi \in [0, 1]$ on the entire range of costs and values $[5, 9]^2$. In line with Theorem 1, trade remains possible on sufficiently small cost and value ranges: as illustrated in Figure 6, near $\gamma = 0$ the size of trading range is roughly linear in γ (left and right subfigures). The figure also shows that as we approached $\gamma = 1$ (the log utility) the trading range quickly grows (central subfigure) and the lower

[24] E.g., using data from a television game, Beetsma and Schotman (2001) estimate γ to be between .42 and 13.08; using lifecycle consumption data, Gourinchas and Parker (2002) estimate γ to be between .5 to 1.4; and using data on labor supply, Chetty (2006) estimates γ at .71 and at .97 depending on assumptions made.

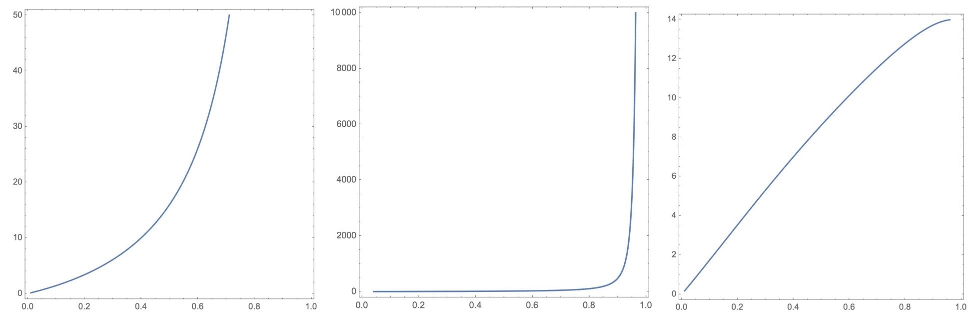

Figure 6: The size x of range $[5, 5+x]^2 \ni (c, v)$ on which efficient trade is possible as function of γ near infinity (left) and near zero (center), and the size x of range $\left[7 - \frac{x}{2}, 7 + \frac{x}{2}\right]^2 \ni (c, v)$ on which efficient trade is possible as function of γ (right).

bound of the range can approach 0 (right subfigure).[25]

5 Ex-post Implementation and Ex-ante Efficiency

So far our analysis has focused on Bayesian implementation. Dominant-strategy and ex-post implementation are more desirable, but are typically much harder to achieve. In our setting, dominant-strategy implementation is possible in cases where efficiency requires winner-take-all randomization. Such cases arise in surprisingly familiar settings, as we demonstrate below.

Theorem 3. *Suppose that for any type θ and any $\lambda \in (0,1)$ we have*

$$\lambda u(1, M, \theta) + (1-\lambda) u(0, 0, \theta) \geq u(1, \lambda M, \theta).$$

Then, efficient trade can be implemented in dominant strategies.

Theorem 3, proven in the appendix, demonstrates that the possibilities for efficient trade in quasilinear settings extend beyond the cases captured by

[25]The flattening of the dependence as we approach $\gamma = 1$ (log utility) in the right-most plot is due to the lower bound of the value and cost ranges, $7 - \frac{x}{2}$, being then close to 0.

Theorem 1 and Theorem 2. Remarkably, in Theorem 3, we do not need to impose any assumptions on the distribution of traders' types. Furthermore, we can fully relax the differentiability, concavity, and Inada condition of the utility function, as well as condition (1). Instead, Theorem 3 relies on the synergy between money and the good captured by the assumption of the theorem.[26] This synergy assumption is satisfied, for instance, when the agent's utility of money that arises when he or she has the good is increasing, at least linearly, in money holdings, $\lambda u(1, M, \theta) \geq u(1, \lambda M, \theta)$, while the utility from no good and no money is nonnegative. This assumption is satisfied by the popular Cobb-Douglas utility function; see Example 3 in Section 6.

In interior cases, where efficiency requires randomization, both dominant-strategy and ex-post implementation are generically impossible in our setting. These ideas are formalized in the following result.

Theorem 4. *When $m^S, m^B \in (0, M)$ and the money endowments are such that efficiency requires randomization, then for a generic utility function u, no mechanism can implement efficient trade in an ex-post equilibrium.*

To establish this result note that $m^S, m^B \in (0, M)$ implies that money transfers are determined by first-order conditions and—when efficiency requires randomization—any mechanism implementing efficiency in ex-post equilibrium assigns the good to the seller with probability π that satisfies (4)-(5). The validity of Theorem 4 then follows from the standard theory of linear partial differential equations, which established that generically there is no solution to the system of equations (4)-(5) if expectations are not taken.[27]

Finally, it is easy to see that ex-ante efficiency cannot be achieved in general:

Theorem 5. *Suppose the range of types is sufficiently large so that either the buyer or the seller can have strictly higher willingness to pay for the good. With separable utility, there is no Bayesian-compatible mechanism that implements ex-ante efficient trade.*

[26] In stark contrast to Theorems 1 and 2, this synergy allows us to implement efficient trade without even eliciting utility types.

[27] Cf. Thomas (1934) and Mardare (2007).

Indeed, in the separable case efficient transfers do not depend on reports; we prove this in Appendix D. At the same time ex-ante efficiency requires that there is a cutoff ratio of the buyer's and the seller's values such that above this ratio the buyer obtains the good with probability 1 and below this ratio the seller does. Taken together, these two facts make it impossible to elicit types, and hence these facts make it impossible to implement the efficient allocation.[28]

6 Examples

We complete our analysis with three examples that illustrate both the possibility of efficient trade and the tractability of the methodology we developed. The discussion of the first two examples focuses on the key steps of the construction of efficient mechanisms we developed in the proof of Theorems 1 and 2, with some of the calculations presented in Appendix A. Example 3 illustrates efficient trade in the environment of Theorem 3.

The trading parties in Examples 1 and 2 have separable utility of the form $u(x, m; \theta) = \theta x + U(m)$. The separable case has several special features that we use in our analysis of these two examples and that we establish in Appendix D. In the optimal contract, each player's money allocation is invariant across states and hence each player receives the same money allocation regardless of whether or not they consume the indivisible good:

$$m^S = M - m^B. \tag{8}$$

The seller's money allocation is implicitly defined by the equation

$$c = \frac{\frac{\partial}{\partial m} U(m^S)}{\frac{\partial}{\partial m} U(M - m^S)} v. \tag{9}$$

The separable utility form and these formulas for money allocations allow us

[28]A similar argument establishes the non-existence of a Bayesian-compatible mechanism that implements ex-ante efficient trade in the interior case of Theorem 3.

to simplify the first-order conditions we derived in Section 3.1 to

$$0 = E_v\{\frac{\partial\pi(c,v)}{\partial c}c + \frac{\partial U(m^S(c,v))}{\partial c}\} \tag{10}$$

and

$$0 = E_c\{-\frac{\partial\pi(c,v)}{\partial v}v + \frac{\partial U(m^B(c,v))}{\partial v}\}. \tag{11}$$

Furthermore, condition (1) is satisfied, which is sufficient for the second-order condition.[29]

Example 1: Log Preferences

Consider trade between a seller and a buyer whose utility of money is logarithmic; log utility is a focal member of the family of CRRA utility functions we discussed in Section 4. Each of the parties privately knows their value of the good being traded. For each of them, utility is separable in money and the good:

$$u(x, m, \theta) = x\theta + \log(m)$$

where $x = 1$ when the agent has the good and 0 otherwise, θ is the privately known value of the good—denoted by c for the seller and v for the buyer—and m is money held. Suppose that c, v are distributed independently and uniformly on $[2, 100]$. Suppose they have combined money holdings of $M = 1$, and their initial money endowments m_b and m_s are such that their utilities are equal if each agent has the mean type $v = c = 51$. Since the seller initially owns the good, this implies that the buyer has most of the money.

We construct the mechanism implementing efficient trade as follows. Our analysis of the efficient frontier—equations (8) and (9)—imply that the efficient

[29] Separability introduces an interesting interpretation—suggested to us by Gregory Pavlov—of the contracts specified in Examples 1 and 2 as examples of a weighted Vickrey-Clarke-Groves (VCG) mechanism. Define the efficient allocation to be that which maximizes the weighted sum of utilities where weights are the reciprocals of the agents' private values. Then, utility is quasilinear in the probability of receiving the good and VCG transfers, expressed in terms of probability units, can be specified that ensure truth-telling and are budget balanced in the sense that probabilities sum to 1. These "transfers" leave each agent with exactly the consumption probabilities determined by our mechanism.

money holdings are

$$m^B(c,v) = \frac{Mc}{c+v} \quad \text{and} \quad m^S(c,v) = \frac{Mv}{c+v},$$

and that these holdings do not depend on which party receives the good. To fully define the trade mechanism it remains to construct the probability $\pi(c,v)$ that the seller keeps the good; the probability the buyer receives the good is then $1-\pi(c,v)$. In Appendix A.1, we follow the procedure from the proof of Theorem 1 and show that

$$\pi(c,v) = \frac{1}{2}+\frac{1}{98}\int_c^{51} \frac{-\log(100+x)+\log(2+x)}{x}dx+\frac{1}{98}\int_v^{51} \frac{\log(100+x)-\log(2+x)}{x}dx$$

takes values in $[0,1]$ and gives us a Bayesian incentive compatible mechanism.

It remains to verify that for any pair of types in the range $[2,100]^2$ the mechanism is individually rational. The expected utility of type c seller participating in the mechanism is $\frac{1}{98}\int_2^{100} \pi(c,v)c + \log(\frac{v}{c+v})dv$. Our choice of money endowments is such that when $v=c=51$, the endowment point lies at the intersection of the two Pareto frontiers that correspond to the case where the seller has the item and the case where the buyer has the item; this places us in the interior solution case (the third case in the proof of Theorem 1) and allows to calculate the initial money endowment of the seller to be $\frac{1}{1+e^{51}}$. The expected net utility gain for the seller of type $c \in [2,100]$ from participating in the mechanism is

$$\frac{1}{98}\int_2^{100} \pi(c,v)c + \log(\frac{v}{c+v})dv - c - \log(\frac{1}{1+e^{51}}).$$

Likewise, the expected net utility gain for the buyer of type $v \in [2,100]$ from participating in the mechanism is

$$\frac{1}{98}\int_2^{100} (1-\pi(c,v))v + \log(\frac{c}{c+v})dc - \log(\frac{e^{51}}{1+e^{51}}).$$

The plots in Figure 7 show that both functions are always strictly positive. In particular, the expected net benefit to the seller at $c=100$ and the buyer at $v=2$ is $0.79384 > 0$.

The mechanism is most beneficial to low seller types and high buyer types.

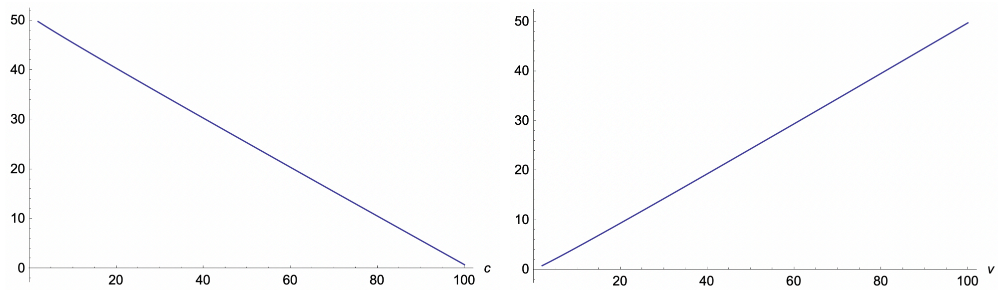

Figure 7: Net utility gain from participating in the mechanism. Left chart shows the gain to the seller for seller types ranging from 2 to 100. Right chart shows the gain to the buyer for buyer types ranging from 2 to 100.

This makes sense since the gains to trade are greatest when the seller value is the lowest and the buyer value is the highest.

Example 2: CARA utility

Consider two traders whose utility of money, $U(m) = \frac{1-e^{-\alpha m}}{\alpha}$, exhibits constant absolute risk aversion. For the sake of the example, suppose that the traders' types c, v are distributed independently and uniformly on $[1, 2]$ and that their money endowments are $m_s = 3.32$ and $m_b = 16.68$.[30]

We construct a mechanism implementing efficient trade—and hence show that such trade is possible—for all $\alpha \geq .2$. For lower α our construction fails and, as α goes to zero, the traders utilities approach the quasi-linear case where efficient trade is not possible.[31] Our analysis of the efficient frontier—equations (8) and (9)—imply that the efficient money holdings are

$$m^B(c, v) = 10 + \frac{\log(c) - \log(v)}{2\alpha} \quad \text{and} \quad m^S(c, v) = 10 + \frac{\log(v) - \log(c)}{2\alpha},$$

and that these holdings do not depend on which party receives the good. As in Example 1, to fully define the trade mechanism it remains to construct the probability $\pi(c, v)$ that the seller keeps the good; the probability the buyer

[30]As in the log example, the chosen money endowments are such that for mean types $c^* = v^* = 1.5$ the pre-trade utility profile can be Pareto improved by trade and efficient trade requires eliciting traders' values.

[31]Our construction fails for $\alpha < .2$ because the values of the probability function $\pi(c, v)$ we construct are then no longer guaranteed to be in $[0, 1]$.

receives the good is then $1-\pi\left(c,v\right)$. In Appendix A.2, we show that

$$\pi(c,v)=1-\frac{e^{-3.32\alpha}-e^{-16.68\alpha}}{3\alpha}+(\sqrt{2}-1)\frac{e^{-10\alpha}}{\alpha}\left[\frac{2}{\sqrt{v}}-\frac{2}{\sqrt{c}}\right]$$

takes values in $[0,1]$ and gives us a Bayesian incentive compatible mechanism.

Appendix A.3 shows that trade improves type-c seller's expected interim utility by

$$\pi(\frac{3}{2},\frac{3}{2})c+\frac{e^{-10\alpha}}{\alpha}\left(4(\sqrt{2}-1)^2c-4(\sqrt{2}-1)\sqrt{c}\right)-c+\frac{e^{-3.32\alpha}}{\alpha},$$

and improves type-v buyer's expected interim utility by

$$(1-\pi^*(\alpha))v+\frac{e^{-10\alpha}}{\alpha}\left(4(\sqrt{2}-1)^2v-4(\sqrt{2}-1)\sqrt{v}\right)+\frac{e^{-16.68\alpha}}{\alpha}.$$

When $\alpha\geq .2$, both these utility gains are positive for all types. As in the log example above, seller's utility gain from trade is strictly decreasing and utility gain is strictly increasing in their respective types.

Example 3: Shifted Cobb-Douglas preferences

In our third example, we illustrate the possibility established in Theorem 3 of implementing efficient trade in dominant strategies. Suppose that the seller's type c and the buyer's type v are distributed according to an arbitrary distribution on $[0,1]^2$. Moreover, each agent has a shifted Cobb-Douglas utility: $u\left(x,m,\theta\right)=\left(1+\theta x\right)m$. The mechanism that generates efficient trade allocates the good and the sum of the money endowments of both agents to the seller with probability $\frac{m_s}{m_s+m_b}$ and it allocates the good and the sum of money endowments to the buyer with the remaining probability $\frac{m_b}{m_s+m_b}$.

This mechanism ϕ is dominant-strategy incentive compatible because the allocation and transfers do not depend on the agents' reports. To see that the mechanism is individually rational, notice that the mechanism gives the seller

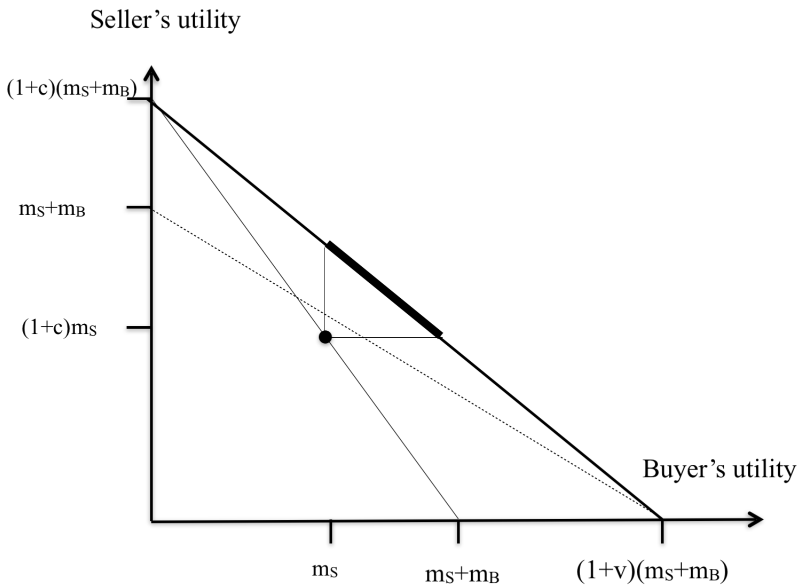

Figure 8: Pareto Frontier for the Shifted Cobb-Douglas example.

with type c an expected utility of

$$\frac{m_s}{m_s + m_b} (1 + c) (m_s + m_b) = (1 + c) m_s$$

and that this expected utility is equal to the utility of the seller if no trade takes place. Similarly, the mechanism gives the buyer with type v an expected utility of

$$\frac{m_b}{m_s + m_b} (1 + v) (m_s + m_b) = (1 + v) m_b$$

and this expected utility is larger than the utility of the buyer if no trade takes place, which is equal to m_b.

Finally, to see that the mechanism is efficient notice that the Pareto frontier in this example consists of all randomizations among two outcomes: either the seller keeps the good and gets all the financial wealth, or the buyer gets the good and all the financial wealth. This can be immediately seen in Figure 8 (for an analytical argument see the proof of Theorem 3 in Appendix F).[32]

[32]Once one has seen this example, it is not surprising that efficient trade is possible

7 Concluding Remarks

We focused on providing incentives for agents to truthfully reveal their cost/value information. It is natural to think that preferences are not observable and need to be elicited, while information such as the size of money holdings can be objectively verified. At the same time, in some environments, for instance in Example 3 of Section 6, we can not only incentivize agents to reveal their value/cost of the good, we can also provide incentives for them to truthfully announce their money holdings, provided the cost of delivering more money than one has (in the event one is asked to do it) is appropriately high. This is so because—as long as the agent is able to deliver the money—each agent benefits from reporting higher money holdings rather than lower.

Our results on efficient trade open the possibility that other problems might have efficient solutions in non-quasilinear settings. For instance, our results imply the possibility of an efficient mechanism for two agents to make a binary decision, e.g. whether to provide a public good, when each of the agents favors a different decision and each has higher marginal utility of money if his preferred decision is taken.

We establish the generic impossibility of achieving efficient trade in an ex-post incentive compatible way in our setting and we demonstrate that in some special cases efficient trade can be accomplished in a strategy-proof way. While we focus on bilateral trade, our analysis can also be used to show that, in the allocation setting, generically no ex-post incentive compatible mechanism is efficient.

in this specific setting. Even though agents have private information about their types, private information does not create any uncertainty about the potential for gains from trade. Whomever consumes the good has the highest (constant) marginal utility of wealth and hence efficiency is achieved by giving whomever consumes the good all of the money. Thus, efficiency is achieved by randomizing over the two extreme allocations identified in Figure 8 and the only burden on the mechanism is to define the probability of exchange in a way that ensures individual rationality. An even simpler example arises when agents' have standard Cobb-Douglas utilities over the good and the money, $u(x, m; \theta) = A(\theta) x^{\alpha(\theta)} m^{\beta(\theta)}$ for some functions A, α, β. With Cobb-Douglas utilities, the mechanism that allocates the good and all the money to the seller implements efficient trade. Of course, such an example is extreme since, without the indivisible good, agents have no use for money.

References

1. Beetsma, Roel M. W. J., and Peter C. Schotman (2001). “Measuring Risk Attitudes in a Natural Experiment: Data from the Television Game Show Lingo.” *The Economic Journal* 111(474), 821–848.

2. Benkert, Jean-Michel (2022). “Bilateral Trade with Loss-Averse Agents.” Working Paper.

3. Blumrosen, Liad, and Shahar Dobzinski (2021). “(Almost) efficient mechanisms for bilateral trading.” *Games and Economic Behavior* 130, 369–383.

4. Campbell, John Y. (2018). *Financial Decisions and Markets: A Course in Asset Pricing.* Princeton University Press

5. Chatterjee, Kalyan, and William Samuelson (1983). “Bargaining under Asymmetric Information.” *Operations Research* 31, 835–851.

6. Chetty, Raj (2006). “A New Method of Estimating Risk Aversion.” *American Economic Review* 96(5), 1821–1834.

7. Coase Ronald (1960). “The Problem of Social Cost.” *Journal of Law and Economics* 3, 1–44.

8. Cook, Philip J. and Daniel A. Graham (1977). “The Demand for Insurance and Protection: The Case of Irreplaceable Commodities.” *Quarterly Journal of Economics* 91, 143-156.

9. Copic, Jernej, and Clara Ponsati (2016). “Optimal robust bilateral trade: Risk neutrality.” *Journal of Economic Theory* 163, 276–287.

10. Cramton, Peter, Robert Gibbons, and Paul Klemperer (1987). “Dissolving a Partnership Efficiently.” *Econometrica* 55(3), 615-632.

11. Crawford, Vincent P. (2021) “Efficient Mechanisms for Level-k Bilateral Trading.” *Games and Economic Behavior* 127, 80-101.

12. Cremer, Jacques and Richard P. McLean (1988). “Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions.” *Econometrica* 56(6), 1247-57.

13. Cripps, Martin. W. and Swinkels, Jeroen. M. (2006). “Efficiency of Large Double Auctions.” *Econometrica* 74(1), 47–92.

14. Demsetz, Harold (1968), “The Cost of Transacting.” *Quarterly Journal of Economics* 82, 33–53.

15. Garratt, Rod (1999). “On Bargaining for an Indivisible Good.” *Games and Economic Behavior* 26, 186-192.

16. Garratt, Rod and John Marshall (1994). “Public Finance of Private Goods: The Case of College Education” *Journal of Political Economy* 102 (3), 566–582.

17. Gourinchas, Pierre-Olivier and Jonathan Parker (2002). “Consumption over the Lifecycle.” *Econometrica* 70(1), 47–91.

18. Gresik, Thomas A., and Mark A. Satterthwaite, 1983. “The Number of Traders Required to Make a Market Competitive: The Beginnings of a Theory.” Discussion Papers 551, Northwestern University, Center for Mathematical Studies in Economics and Management Science.

19. Holt, Charles. 1980. “Competitive Bidding for Contracts under Alternative Auction Procedures,” *Journal of Political Economy* 88(3), 433–45.

20. Holt, Charles A., and Susan K. Laury (2002). “Risk Aversion and Incentive Effects.” *American Economic Review* 92(5), 1644–1655.

21. Jantschgi, Simon, Heinrich H. Nax, Bary S. R. Pradelski, and Marek Pycia (2022). “Markets and Transaction Costs.” Working Paper.

22. Kazumura, Tomoya, Debasis Mishra, and Shigehiro Serizawa (2020). “Mechanism Design without Quasilinearity.” *Theoretical Economics* 15, 511-544.

23. Krishna, Vijay and Motty Perry (1997). "Efficient Mechanism Design." Working paper, Penn State University.

24. Loertscher, Simon, Leslie M. Marx, and Tom Wilkening, (2015). "A Long Way Coming: Designing Centralized Markets with Privately Informed Buyers and Sellers." *Journal of Economic Literature* 53(4), 857-97.

25. Makowski, Louis, and Joseph M. Ostroy, 1989. "Efficient and Individually Rational Bayesian Mechanisms Only Exist on Perfectly Competitive Environments." *UCLA Economics Working Papers 566*, UCLA Department of Economics.

26. Mardare, Sorin (2007) "On Systems of First Order Linear Partial Differential Equations with L^p Coefficients." *Advances in Differential Equations* 12, 301-360.

27. Maskin, Eric and John Riley (1984). "Optimal Auctions with Risk Averse Buyers." *Econometrica* 52(6), 1473-1518.

28. Matthews, Steven. 1983. "Selling to Risk Averse Buyers with Unobservable Tastes," *Journal of Economic Theory* 30, 370–400.

29. Matsuo, Toshihide (1989): "On Incentive Compatibility, Individual Rationality, and Ex Post Efficient Mechanisms for Bilateral Trading", *Journal of Economic Theory*, 49, 189-194.

30. McAfee, R. Preston and Philip J. Reny (1992). "Correlated Information and Mechanism Design." *Econometrica* 60(2) 395-421.

31. McAfee, R. Preston (1991). "Efficient Allocation with Continuous Quantities." *Journal of Economic Theory* 53 (1), 51-74.

32. Milgrom, Paul (2004): *Putting Auction Theory to Work.* Cambridge University Press.

33. Myerson, Roger B. and Mark A. Satterthwaite (1983). “Efficient Mechanisms for Bilateral Trading." *Journal of Economic Theory* 29(2), 265-281.

34. Reny, Philip J. and Motty Perry (2006). “Toward a Strategic Foundation for Rational Expectations Equilibrium.” *Econometrica* 74: 1231–1269.

35. Riley, John (2012). *Essential Microeconomics.* Cambridge University Press.

36. Rogerson, Richard (1988). “Indivisible labor, lotteries and equilibrium.” *Journal of Monetary Economics* 21, 3–16.

37. Rustichini, Aldo, Mark A. Satterthwaite and Steven R. Williams (1994) “Convergence to Efficiency in a Simple Market with Incomplete Information.” *Econometrica* 62(1), 1041–1063.

38. Simonovska, Ina, and Michael E. Waugh (2014). “The Elasticity of Trade: Estimates and Evidence.” *Journal of International Economics* 92, 34-50.

39. Thomas, Tracey Y. (1934). “Systems of Total Differential Equations Defined Over Simply Connected Domains.” *Annals of Mathematics* 35 (4), 730-734.

40. Weitzman, Martin L. (1977). “Is the price system or rationing more effective in getting a commodity to those who need it most?” *The Bell Journal of Economics* 8, 517–524.

41. Williams, Steven R. (1999). “A Characterization of Efficient Bayesian Incentive Compatible Mechanisms." *Economic Theory* 14, 155-180.

42. Williamson, Oliver E. (2010). “Transaction Cost Economics: The Natural Progression.” *American Economic Review* 100(3), 673-90.

43. Wilson, Robert (1985). “Incentive Efficiency of Double Auctions.” *Econometrica* 53, 1101–1115.

44. Wolitzky, Alexander (2016). “Mechanism Design with Maxmin Agents: Theory and an Application to Bilateral Trade.” *Theoretical Economics* 11, 971-1004.

A Examples 1 and 2: Constructing Probability π

Our analysis of Theorem 1 shows that any probability function π solving the first-order conditions (10) and (11) determines a Bayesian incentive compatible mechanism, because the second-order condition assumed in our analysis is always satisfied when agents' utilities are separable in the good and money.

A.1 Construction of incentive compatible $\pi(c, v)$ in Example 1

Plugging in the formulas for money-holding from the example simplifies the first-order conditions (10) and (11), to:

$$E_v\left[-\frac{\partial\pi(c,v)}{\partial c}c\right] = E_v\left[-\frac{1}{c+v}\right] \tag{12}$$

and

$$E_c\left[-\frac{\partial\pi(c,v)}{\partial v}v\right] = E_c\left[\frac{1}{c+v}\right]. \tag{13}$$

By following the procedure we developed in the proof of Lemma 1, we find a solution of this systems of partial differential equations of the form $\pi(c, v) = b(v)\,\Delta^b(c, v) + s(c)\,\Delta^s(c, v)$. As in the lemma, we can set $\Delta^s(c, v)$ to be any solution of the homogenous version of (12) and we can set $\Delta^b(c, v)$ to be any solution of the homogenous version of (13). In the separable case, one pair of solutions is always

$$\Delta^s(c, v) = 1, \quad \Delta^b(c, v) = 1,$$

which is a major simplification of the general case. The right-hand side of (12) is

$$\phi\left(c\right)=E_{v}\left[-\frac{1}{c+v}\right]=-\int_{2}^{100}\frac{1}{c+v}\frac{1}{98}dv=\frac{-\log\left(100+c\right)+\log\left(2+c\right)}{98},$$

and the right-hand side of (13) is

$$\psi\left(v\right)=-E_{c}\left[-\frac{1}{c+v}\right]=\int_{2}^{100}\frac{1}{c+v}\frac{1}{98}dc=\frac{\log\left(100+v\right)-\log\left(2+v\right)}{98}.$$

The lemma procedure allows us to set the auxiliary function $b\left(\cdot\right)$ to be any solution of the ordinary differential equation (ODE)

$$\begin{aligned}\psi\left(v\right)&=E_{c}\left[B_{1}\left(c,v\right)\Delta^{b}\left(c,v\right)\right]b'\left(v\right)+E_{c}\left[B_{1}\left(c,v\right)\frac{\partial}{\partial v}\Delta^{b}\left(c,v\right)+B_{2}\left(c,v\right)\Delta^{b}\left(c,v\right)\right]b\left(v\right)\\&=E_{c}\left[v\right]b'\left(v\right)=vb'\left(v\right),\end{aligned}$$

and thus

$$b'\left(v\right)=\frac{\log\left(100+v\right)-\log\left(2+v\right)}{98v}.$$

Similarly, the equation for s is $\phi\left(c\right)=cs'\left(c\right)$, and thus

$$s'\left(c\right)=\frac{-\log\left(100+c\right)+\log\left(2+c\right)}{98c}.$$

As $\Delta^{s}=\Delta^{b}=1$, the variations in the seller's report adjust the probability by the amount

$$\int_{c}^{51}s'(x)dx=\frac{1}{98}\int_{c}^{51}\frac{-\log(100+x)+\log(2+x)}{x}dx,$$

and the variations in the buyer's report adjust the probability by the amount

$$\int_{v}^{51}b'(x)dx=\frac{1}{98}\int_{v}^{51}\frac{\log(100+x)-\log(2+x)}{x}dx.$$

Setting the anchoring probability at $\pi^{*}=.5$—that is assuming that the seller retains the item with probability .5 when the reports are $51, 51$—we obtain that the probability that the seller gets the item if the seller reports c and the buyer reports v is

$$\pi(c,v) = \frac{1}{2}+\frac{1}{98}\int_c^{51}\frac{-\log(100+x)+\log(2+x)}{x}dx+\frac{1}{98}\int_v^{51}\frac{\log(100+x)-\log(2+x)}{x}dx.$$

The probability π is increasing in c and decreasing in v, and hence to verify that $\pi(c,v)\in[0,1]$, it is enough to verify that $\pi(2,100), \pi(100,2)$ are in this interval; they are. The proposed mechanism is thus well defined and, by construction, incentive compatible.

A.2 Construction of incentive compatible $\pi(c,v)$ in Example 2

Plugging in the formulas for money-holding from the example simplifies the first-order conditions (10) and (11), to:

$$0 = E_v\left[\frac{\partial\pi(c,v)}{\partial c}c - e^{-10\alpha}\frac{1}{2\sqrt{c}\sqrt{v}\alpha}\right]$$

and

$$0 = E_c\left[-\frac{\partial\pi(c,v)}{\partial v}v - e^{-10\alpha}\frac{1}{2\sqrt{c}\sqrt{v}\alpha}\right].$$

As in Example 1, we solve these equations by following the constructive procedure from the constructive proof of Lemma 1. Define

$$\psi(v) = -E_c\left[-e^{-10\alpha}\frac{1}{2\sqrt{c}\sqrt{v}\alpha}\right] = \frac{1}{\alpha}\int_1^2 e^{-10\alpha}\frac{1}{2\sqrt{c}\sqrt{v}}dc = \frac{1}{\alpha}e^{-10\alpha}\frac{\sqrt{2}-1}{\sqrt{v}}$$

and

$$\phi(c) = E_v\left[-e^{-10\alpha}\frac{1}{2\sqrt{c}\sqrt{v}\alpha}\right] = -\frac{1}{\alpha}\int_1^2 e^{-10\alpha}\frac{1}{2\sqrt{c}\sqrt{v}}dv = \frac{1}{\alpha}e^{-10\alpha}\frac{1-\sqrt{2}}{\sqrt{c}}.$$

Following the lemma procedure, we set

$$\Delta^s(c,v) = 1, \quad \Delta^b(c,v) = 1.$$

The functions $b(\cdot)$ and $s(\cdot)$ are given by the ODEs $\psi(v) = vb'(v)$ and $\phi(c) = cs'(c)$, respectively. Thus, $b'(v) = \frac{1}{\alpha}e^{-10\alpha}\frac{\sqrt{2}-1}{v^{\frac{3}{2}}}$ and $s'(c) = \frac{1}{\alpha}e^{-10\alpha}\frac{1-\sqrt{2}}{c^{\frac{3}{2}}}$, and the seller's report adjusts the probability π by the amount $\int_c^{1.5} s'(x)dx = -(\sqrt{2}-1)\frac{e^{-10\alpha}}{\alpha}\int_c^{1.5} x^{-\frac{3}{2}}dx$ while the buyer's report adjusts it by the amount $\int_v^{1.5} b'(x)dx = (\sqrt{2}-1)\frac{e^{-10\alpha}}{\alpha}\int_v^{1.5} x^{-\frac{3}{2}}dx$. In effect, the probability that the seller gets the item if the seller reports c and the buyer reports v is

$$\begin{aligned}
\pi(c,v) &= \pi^*(\alpha) + (\sqrt{2}-1)\frac{e^{-10\alpha}}{\alpha}\left[\int_v^{1.5} x^{-\frac{3}{2}}dx - \int_c^{1.5} x^{-\frac{3}{2}}dx\right] \\
&= \pi^*(\alpha) + (\sqrt{2}-1)\frac{e^{-10\alpha}}{\alpha}\int_v^{c} x^{-\frac{3}{2}}dx \\
&= \pi^*(\alpha) + (\sqrt{2}-1)\frac{e^{-10\alpha}}{\alpha}\left[\frac{2}{\sqrt{v}} - \frac{2}{\sqrt{c}}\right],
\end{aligned}$$

for some anchoring value $\pi^*(\alpha)$. In the log example at this point we set $\pi^*(\alpha)$ to be $\frac{1}{2}$ but in the present example we set this anchoring probability value to be the following function of α:

$$\pi^*(\alpha) = 1 - \frac{e^{-3.32\alpha} - e^{-16.68\alpha}}{3\alpha}.$$

For the resulting mechanism to be well defined we need to show that this choice of $\pi^*(\alpha)$ leads to π being a probability; the mechanism is then incentive compatible by construction.

To verify that $\pi(c,v) \in [0,1]$, start by noticing that π is increasing in c and decreasing in v. The verification thus reduces to ensuring that

$$0 \leq \pi(1,2) = \pi^*(\alpha) + (\sqrt{2}-1)\frac{e^{-10\alpha}}{\alpha}\left[\frac{2}{\sqrt{2}} - \frac{2}{\sqrt{1}}\right],$$

and

$$1 \geq \pi(2,1) = \pi^*(\alpha) + (\sqrt{2}-1)\frac{e^{-10\alpha}}{\alpha}\left[\frac{2}{\sqrt{1}} - \frac{2}{\sqrt{2}}\right].$$

The resulting restriction

$$\pi^*(\alpha) \in \left[\frac{e^{-10\alpha}}{\alpha}\left(\sqrt{2}-1\right)\left(2-\sqrt{2}\right), 1 - \frac{e^{-10\alpha}}{\alpha}\left(\sqrt{2}-1\right)\left(2-\sqrt{2}\right)\right]$$

is indeed satisfied by the above choice of $\pi^*(\alpha)$ for any $\alpha \geq .2$.[33]

A.3 Verifying Interim Individual Rationality in Example 2

Interim individual rationality requires that any type of the buyer and any type of the seller are better off under the mechanism than under no trade, under the maintained assumption that $\alpha \geq .2$. The no-trade payoff of type-c seller is $c + \frac{1-e^{-3.32\alpha}}{\alpha}$ and the expected payoff from the mechanism is

$$\begin{aligned}
& \int_1^2 \pi(c,v;\alpha)c + \frac{1}{\alpha} - \frac{e^{-10\alpha}}{\alpha}\frac{\sqrt{c}}{\sqrt{v}}dv \\
= \; & \frac{1}{\alpha} + \left(\pi^*(\alpha) + 4(\sqrt{2}-1)^2\frac{e^{-10\alpha}}{\alpha}\right)c - 4(\sqrt{2}-1)\frac{e^{-10\alpha}}{\alpha}\sqrt{c},
\end{aligned}$$

giving the expected net utility gain of

$$\pi^*(\alpha)c + \frac{e^{-10\alpha}}{\alpha}\left(4(\sqrt{2}-1)^2 c - 4(\sqrt{2}-1)\sqrt{c}\right) - c + \frac{e^{-3.32\alpha}}{\alpha}.$$

For $\alpha \geq .2$, this net gain is strictly decreasing in c over the interval $[1,2]$ and achieves the minimum at $c = 2$. Thus, the seller's participation constraint is satisfied when

$$2(\pi^*(\alpha)-1) - 4(\sqrt{2}-1)(\sqrt{2}-2)\frac{e^{-10\alpha}}{\alpha} + \frac{e^{-3.32\alpha}}{\alpha} \geq 0.$$

[33] The choice of $\pi^*(\alpha)$ is made so as to make the minimum gain of the seller equal the minimum gain of the buyer; both gains calculated in Section 6. The restriction to $\alpha \geq .2$ is then needed to ensure that $\pi(c,v) \in [0,1]$.

With $\pi^*(\alpha)$ defined as above, this restriction reduces to

$$\frac{e^{-3.32\alpha} - e^{-16.68\alpha}}{3\alpha} - 4(\sqrt{2}-1)(\sqrt{2}-2)\frac{e^{-10\alpha}}{\alpha} + \frac{e^{-16.68\alpha}}{\alpha} \geq 0,$$

and it is satisfied for $\alpha \geq .2$.[34]

Similarly, the no-trade payoff of type-v buyer is $\frac{1-e^{-16.68\alpha}}{\alpha}$ and the expected payoff from the mechanism is

$$\begin{aligned} & \int_1^2 (1-\pi(c,v;\alpha))v + \frac{1}{\alpha} - \frac{e^{-10\alpha}}{\alpha}\frac{\sqrt{v}}{\sqrt{c}}dc \\ = \; & \frac{1}{\alpha} + \left((1-\pi^*(\alpha)) + 4(\sqrt{2}-1)^2\frac{e^{-10\alpha}}{\alpha}\right)v - 4(\sqrt{2}-1)\frac{e^{-10\alpha}}{\alpha}\sqrt{v}. \end{aligned}$$

It follows that the expected net utility gain is

$$(1-\pi^*(\alpha))v + \frac{e^{-10\alpha}}{\alpha}\left(4(\sqrt{2}-1)^2 v - 4(\sqrt{2}-1)\sqrt{v}\right) + \frac{e^{-16.68\alpha}}{\alpha}.$$

For $\alpha \geq .2$, this net gain is strictly increasing in v over $[1,2]$ (due to our choice of $\pi^*(\alpha)$) and achieves a minimum at $v = 1$. Hence the buyer's participation constraint is satisfied when

$$(1-\pi^*(\alpha)) - 4(\sqrt{2}-1)(\sqrt{2}-2)\frac{e^{-10\alpha}}{\alpha} + \frac{e^{-16.68\alpha}}{\alpha} \geq 0.$$

With $\pi^*(\alpha)$ defined as above, this restriction reduces to the same formula as the seller's participation constraint and it is satisfied for $\alpha \geq .2$.

[34] While the validity of the formula is restricted to $\alpha \geq .2$, the formula itself is nonnegative for all $\alpha > 0$; similar comment is valid for the formula on buyer's minimum net gain.

Omitted Proofs for "Efficient Bilateral Trade"

Appendices For Online Publication

Rodney Garratt and Marek Pycia

B Derivation of Inequalities (3)

Here we derive the inequalities that play a crucial role in our analysis of the second-order condition. We know from (2) that

$$\frac{\frac{\partial}{\partial m} u\left(1, m^S(c,v), c\right)}{u\left(1, m^S(c,v), c\right) - u\left(0, M - m^B(c,v), c\right)} = \frac{\frac{\partial}{\partial m} u\left(0, M - m^S(c,v), v\right)}{u\left(1, m^B(c,v), v\right) - u\left(0, M - m^S(c,v), v\right)}.$$

Let us show that for $m' \leq m$ (notice that $M - m^B(c,v) \leq m^S(c,v)$), the expression $\frac{\frac{\partial}{\partial m} u(1,m,c)}{u(1,m,c) - u(0,m',c)}$ strictly decreases in c. To prove this it is enough to show that

$$\frac{u_c\left(1, m, c\right) - u_c\left(0, m', c\right)}{u\left(1, m, c\right) - u\left(0, m', c\right)} > \frac{u_{cm}\left(1, m, c\right)}{u_m\left(1, m, c\right)} \tag{14}$$

for $m' \leq m$. Let us rewrite the left-hand side of (14) as

$$\begin{aligned}
\frac{u_c\left(1, m, c\right) - u_c\left(0, m', c\right)}{u\left(1, m, c\right) - u\left(0, m', c\right)} &= \frac{\left[u_c\left(1, m, c\right) - u_c\left(1, m', c\right)\right] + \left[u_c\left(1, m', c\right) - u_c\left(0, m', c\right)\right]}{\left[u\left(1, m, c\right) - u\left(1, m', c\right)\right] + \left[u\left(1, m', c\right) - u\left(0, m', c\right)\right]} \\
&= \frac{\left[u_c\left(1, m', c\right) - u_c\left(0, m', c\right)\right] + \int_{m'}^{m} u_{cm}\left(1, \tilde{m}, c\right)}{\left[u\left(1, m', c\right) - u\left(0, m', c\right)\right] + \int_{m'}^{m} u_m\left(1, \tilde{m}, c\right)}.
\end{aligned}$$

As in the analysis of the second-order condition in Section 3.1.2, we use the facts that the inequality in (1) implies

$$\frac{u_c\left(1, m', c\right) - u_c\left(0, m', c\right)}{u\left(1, m', c\right) - u\left(0, m', c\right)} > \frac{\frac{\partial}{\partial m} u_c\left(1, m', c\right)}{\frac{\partial}{\partial m} u\left(1, m', c\right)} \tag{15}$$

and the equality in (1) implies

$$\frac{\frac{\partial}{\partial m} u_c\left(1, \tilde{m}, c\right)}{\frac{\partial}{\partial m} u\left(1, \tilde{m}, c\right)} = \frac{\frac{\partial}{\partial m} u_c\left(1, m', c\right)}{\frac{\partial}{\partial m} u\left(1, m', c\right)} = \frac{\frac{\partial}{\partial m} u_c\left(1, m, c\right)}{\frac{\partial}{\partial m} u\left(1, m, c\right)}. \tag{16}$$

The inequality in (15) and the first equality in (16) establish that

$$\frac{u_c(1,m,c)-u_c(0,m',c)}{u(1,m,c)-u(0,m',c)} > \frac{u_{cm}(1,m',c)}{u_m(1,m',c)}$$

and then, by the second equality in (16), we have that (14) holds. Hence, $\frac{\frac{\partial}{\partial m}u(1,m,c)}{u(1,m,c)-u(0,m',c)}$ strictly decreases in c.

Going back to the first equation of this Appendix, as we increase c while keeping money levels constant the left-hand side decreases, while the right hand side stays constant. Looking at the Figure 1 shows that to balance this out, the critical point shifts towards higher utility of the buyer. And, thus the seller's money level m^S decreases in this critical point. The analysis of other critical money levels is similar.

C Proof of Theorem 1: Steps Omitted in Main Text

C.1 Verifying the Second-Order Conditions

We check the second-order condition for the seller; the buyer's problem is analogous. Since at points at which the first-order condition is satisfied we have

$$0=\frac{d}{dc}\left(\frac{\partial}{\partial\hat{c}}\Pi^S(c,c)\right)=\frac{\partial}{\partial c}\left(\frac{\partial}{\partial\hat{c}}\Pi^S(c,c)\right)+\frac{\partial}{\partial\hat{c}}\left(\frac{\partial}{\partial\hat{c}}\Pi^S(c,c)\right),$$

the second-order condition for the seller is implied if we show that

$$\frac{\partial}{\partial c}\frac{\partial}{\partial\hat{c}}\Pi^S(c,c)>0.$$

A straightforward calculation shows that $\frac{\partial}{\partial c}\frac{\partial}{\partial \hat{c}}\Pi^S(c,c)$ equals

$$\begin{aligned}
&E_v\Bigg\{\left[\frac{\partial}{\partial c}\pi(c,v)\right]\left[u_c\left(1,m^S(c,v),c\right)-u_c\left(0,M-m^B(c,v),c\right)\right]\\
+\quad&\pi(c,v)\left(\left[\frac{\partial}{\partial m}u_c\left(1,m^S(c,v),c\right)\right]\left[\frac{\partial}{\partial c}m^S(c,v)\right]+\left[\frac{\partial}{\partial m}u_c\left(0,M-m^B(c,v),c\right)\right]\left[\frac{\partial}{\partial c}m^B(c,v)\right]\right)\\
-\quad&\left[\frac{\partial}{\partial m}u_c\left(0,M-m^B(c,v),c\right)\right]\left[\frac{\partial}{\partial c}m^B(c,v)\right]\Bigg\}.
\end{aligned}$$

We can substitute in for $\frac{\partial}{\partial c}\pi(c,v)$ from the first-order condition obtaining that $\frac{\partial}{\partial c}\frac{\partial}{\partial \hat{c}}\Pi^S(c,c)$ is equal to $(1-\pi(c,v))\frac{\partial}{\partial c}m^B(c,v)$ times

$$\begin{aligned}
&\left[\frac{\partial}{\partial m}u\left(0,M-m^B(c,v),c\right)\right]\left[u_c\left(1,m^S(c,v),c\right)-u_c\left(0,M-m^B(c,v),c\right)\right]\\
-&\left[\frac{\partial}{\partial m}u_c\left(0,M-m^B(c,v),c\right)\right]\left[u\left(1,m^S(c,v),c\right)-u\left(0,M-m^B(c,v),c\right)\right]
\end{aligned}$$

minus $\pi(c,v)\frac{\partial}{\partial c}m^S(c,v)$ times

$$\begin{aligned}
&\left[\frac{\partial}{\partial m}u\left(1,m^S(c,v),c\right)\right]\left[u_c\left(1,m^S(c,v),c\right)-u_c\left(0,M-m^B(c,v),c\right)\right]\\
-&\left[\frac{\partial}{\partial m}u_c\left(1,m^S(c,v),c\right)\right]\left[u\left(1,m^S(c,v),c\right)-u\left(0,M-m^B(c,v),c\right)\right].
\end{aligned}$$

By (3), $\frac{\partial}{\partial c}m^B(c,v)>0>\frac{\partial}{\partial c}m^S(c,v)$, and thus, for $\pi(c,v)\in(0,1)$, the second-order condition is satisfied provided both above displayed expressions are strictly positive. Since $m^S(c,v)\geq M-m^B(c,v)$, the expressions are positive if

$$\left[\frac{\partial}{\partial m}u(0,m',c)\right]\left[u_c(1,m,c)-u_c(0,m',c)\right]-\left[\frac{\partial}{\partial m}u_c(0,m',c)\right]\left[u(1,m,c)-u(0,m',c)\right]>0$$

and

$$\left[\frac{\partial}{\partial m}u(1,m,c)\right]\left[u_c(1,m,c)-u_c(0,m',c)\right]-\left[\frac{\partial}{\partial m}u_c(1,m,c)\right]\left[u(1,m,c)-u(0,m',c)\right]>0$$

for all $m' \leq m$. The second of these two inequalities has been proven in Appendix B as inequality (14) and the proof of the first of these inequalities follows the same steps as the proof of (14).

C.2 Verification that the first-order conditions in Section 3.1 satisfy the assumptions of Lemma 1

To ensure that the coefficient in front of $\frac{\partial}{\partial c}\pi(c,v)$ is positive, we multiply equation (5) by -1. Then, equations (4) and (5) take the form

$$\begin{aligned} E_v\left[S_1(c,v)\frac{\partial}{\partial c}\pi(c,v) + S_2(c,v)\pi(c,v)\right] &= \phi(c) \\ E_c\left[B_1(c,v)\frac{\partial}{\partial c}\pi(c,v) + B_2(c,v)\pi(c,v)\right] &= \psi(v) \end{aligned}$$

where the coefficients in front of $\frac{\partial}{\partial c}\pi$ and $\frac{\partial}{\partial v}\pi$ are

$$\begin{aligned} S_1(c,v) = u\left(1, m^S(c,v), c\right) - u\left(0, M - m^B(c,v), c\right) &> 0 \\ B_1(c,v) = u\left(1, m^B(c,v), v\right) - u\left(0, M - m^S(c,v), v\right) &> 0 \end{aligned}$$

the coefficients in front of π are

$$\begin{aligned} S_2(c,v) &= \left[\frac{\partial}{\partial m}u\left(1, m^S(c,v), c\right)\right]\left[\frac{\partial}{\partial c}m^S(c,v)\right] + \left[\frac{\partial}{\partial m}u\left(0, M - m^B(c,v), c\right)\right]\left[\frac{\partial}{\partial c}m^B(c,v)\right] \\ B_2(c,v) &= \left[\frac{\partial}{\partial m}u\left(1, m^B(c,v), v\right)\right]\left[\frac{\partial}{\partial v}m^B(c,v)\right] + \left[\frac{\partial}{\partial m}u\left(0, M - m^S(c,v), v\right)\right]\left[\frac{\partial}{\partial v}m^S(c,v)\right] \end{aligned}$$

and the functions ϕ, ψ are given by

$$\begin{aligned} \phi(c) &= E_v\left\{\left[\frac{\partial}{\partial m}u\left(0, M - m^B(c,v), c\right)\right]\left[\frac{\partial}{\partial c}m^B(c,v)\right]\right\} \\ \psi(v) &= -E_c\left\{\left[\frac{\partial}{\partial m}u\left(1, m^B(c,v), v\right)\right]\left[\frac{\partial}{\partial v}m^B(c,v)\right]\right\}. \end{aligned}$$

By assumption, u and its derivatives are continuously differentiable. The continuous differentiability of m^S and m^B—which are implicitly defined in (2)—

follows from the the Implicit Function Theorem and the strict concavity of u.

C.3 Bounding money holding type-dependence and FOCs coefficients in terms of Δ

The last step of the proof of Theorem 1 relies on the claim that by taking sufficiently small Δ we can ensure that the absolute value of the integrals of $\frac{S_2}{S_1}$, $\frac{B_2}{B_1}$, $\frac{\phi}{S_1}$, and $\frac{\psi}{B_1}$ over any interval of length at most L is small. For small Δ, the functions S_1 and B_1 defined in Appendix C.2 take values close to $S_1(c^*, v^*)$ and $B_1(c^*, v^*)$, respectively, and hence are uniformly bounded away from 0. By taking Δ sufficiently small we thus want to ensure that integrals over S_2, B_2, ϕ, and ψ can be made arbitrarily close to 0.

An inspection of the formulas for these four functions shows that it is enough to notice that (a) the partials of u with respect to m (e.g., $\frac{\partial}{\partial m}u$) are bounded uniformly in (c, v), and (b) for any $\varepsilon > 0$ there is $\Delta > 0$ such that if the asymmetry of information is bounded by Δ, then, keeping fixed one type (say c), the integral of the absolute value of partials of m^B and m^S with respect to the other type (say v, e.g., $\frac{\partial}{\partial v}m^B(c, v)$) over any interval of the other type (here v) of length L is bounded by ε. Claim (a) follows because u is twice continuously differentiable in money and type, the space $[0, M]$ of possible money holdings and the space of types (c, v) are both compact and the continuity of m^B implies that $\frac{\partial}{\partial m}u\left(1, m^B(c, v), v\right)$ is bounded uniformly in (c, v).

To prove claim (b), we focus on $\frac{\partial}{\partial v}m^B(c, v)$; the argument for other partials is analogous. Recall that the money levels $m^S(c, v)$ and $m^B(c, v)$ are uniquely determined by equations (2). From the Implicit Function Theorem's expressions on the partials of m^S and m^B with respect to c and v, we now infer that for any $\varepsilon > 0$ there exists $\Delta > 0$ such that with utilities u within Δ of $u(x, m; c^*)$ and $u(x, m; v^*)$ the critical money levels m^S and m^B are within ε of $m^S(c^*, v^*)$ and $m^B(c^*, v^*)$, respectively.

Multiplying equations (2) by the relevant denominators gives us the Jacobian

$$J = \begin{bmatrix} \frac{\partial F}{\partial m^S} & \frac{\partial F}{\partial m^B} \\ \frac{\partial G}{\partial m^S} & \frac{\partial G}{\partial m^B} \end{bmatrix}$$

where $F(m^S, m^B)$ and $G(m^S, m^B)$ are defined as follows (here, u' is the derivative of u with respect to m):

$$F(m^S, m^B) = u'(1, m^S, c)[u(1, m^B, v) - u(0, M - m^S, v)] - u'(0, M - m^S, v)[u(1, m^S, c) - u(0, M - m^B, c)],$$

$$G(m^S, m^B) = u'(1, m^S, c)u'(1, m^B, v) - u'(0, M - m^B, c)u'(0, M - m^S, v).$$

We first show that the Jacobian J has a strictly positive determinant. Introduce the short-hand notation $X = u(1, m^B, v) - u(0, M - m^S, v) > 0$ and $Y = u(1, m^S, c) - u(0, M - m^B, c) > 0$. Then:

$$\frac{\partial F}{\partial m^S} = u''(1, m^S, c)X + u'(1, m^S, c)u'(0, M - m^S, v) + u''(0, M - m^S, v)Y - u'(0, M - m^S, v)u'(1, m^S, c)$$

$$\frac{\partial F}{\partial m^B} = u'(1, m^S, c)u'(1, m^B, v) - u'(0, M - m^S, v)u'(0, M - m^B, c)$$

$$\frac{\partial G}{\partial m^S} = u''(1, m^S, c)u'(1, m^B, v) - u'(0, M - m^B, c)u''(0, M - m^S, v)$$

$$\frac{\partial G}{\partial m^B} = u'(1, m^S, c)u''(1, m^B, v) - u''(0, M - m^B, c)u'(0, M - m^S, v).$$

Recognizing that $u'(1, m^S, c)u'(0, M - m^S, v) - u'(0, M - m^S, v)u'(1, m^S, c) = 0$ in the multiplication of $\frac{\partial F}{\partial m^S}$ by $\frac{\partial G}{\partial m^B}$, we can write down the determinant as:

$$\begin{aligned} D = {} & u''(1, m^S, c)u''(1, m^S, v)u'(1, m^S, c)X + u''(1, m^S, c)u''(0, M - m^B, c)u'(0, M - m^S, v)X \\ & + u''(0, M - m^S, v)u''(1, m^B, v)u'(1, m^S, c)Y + u''(0, M - m^B, c)u''(0, M - m^S, v)u'(0, M - m^S, v)Y \\ & - u''(1, m^s, c)u'(1, m^S, c)(u'(1, m^B, v))^2 + u''(1, m^S, c)u'(1, m^B, v)u'(0, M - m^S, v)u'(0, M - m^B, c) \\ & - u'(1, m^S, c)u'(1, m^B, v)u'(0, M - m^B, c)u''(0, M - m^S, v) \\ & + u''(0, M - m^S, v)u'(0, M - m^B, c)u'(0, M - m^S, v)u'(0, M - m^B, c). \end{aligned}$$

At the optimum $G(m^S, m^B) = 0$, and the last two lines of D are

$$u''(0, M - m^S, v)u'(0, M - m^B, c)[u'(0, M - m^S, v)u'(0, M - m^B, c) - u'(1, m^S, c)u'(1, m^B, v)] = 0.$$

Likewise, the third line of D is

$$u''(1, m^S, c)u'(1, m^B, v)[u'(0, M - m^S, v)u'(0, M - m^B, c) - u'(1, m^S, c)u'(1, m^B, v)] = 0.$$

Collecting terms in the first two lines gives

$$\begin{aligned} D =& u''(1, m^S, c)X[u''(1, m^B, v)u'(1, m^S, c) + u''(0, M - m^B, c)u'(0, M - m^S, v)] \\ &+ u''(0, M - m^S, v)Y[u''(1, m^B, v)u'(1, m^S, c) + u''(0, M - m^B, c)u'(0, M - m^S, v)] \\ =&[u''(1, m^S, c)X + u''(0, M - m^S, v)Y][u''(1, m^B, v)u'(1, m^S, c) + u''(0, M - m^B, c)u'(0, M - m^S, v)]. \end{aligned}$$

The terms in both sets of square brackets are strictly less than zero. Hence, the determinant of the Jacobian is strictly positive.

Given the compactness of the space of arguments and twice continuous differentiability of u, we infer that the infimum of the determinant $\underline{D}$ is also strictly positive. We can thus use the Implicit Function Theorem to compute the partial derivatives $\frac{\partial m^S}{\partial c}$ and $\frac{\partial m^B}{\partial c}$ ($\frac{\partial m^S}{\partial v}$ and $\frac{\partial m^B}{\partial v}$ are computed similarly). We have

$$\begin{bmatrix} \frac{\partial m^S}{\partial c} \\ \frac{\partial m^B}{\partial c} \end{bmatrix} = -\frac{1}{D} \begin{bmatrix} \frac{\partial G}{\partial m^B} & -\frac{\partial F}{\partial m^B} \\ -\frac{\partial G}{\partial m^S} & \frac{\partial F}{\partial m^S} \end{bmatrix} \begin{bmatrix} \frac{\partial F}{\partial c} \\ \frac{\partial C}{\partial c} \end{bmatrix}.$$

In particular, $\frac{\partial m^S}{\partial c} = -\frac{1}{D}\left[\frac{\partial G}{\partial m^B}\frac{\partial F}{\partial c} - \frac{\partial F}{\partial m^B}\frac{\partial G}{\partial c}\right]$. Because $D \geq \underline{D} > 0$, in order to show that the integral of $\frac{\partial m^S}{\partial c}$ over an integral of types c is small it is enough to show that by taking Δ small we can ensure that the corresponding integral of $\frac{\partial F}{\partial c}$ is small and that $\frac{\partial G}{\partial m^B}$ is uniformly bounded (and the analogue for $\frac{\partial F}{\partial m^B}$ and $\frac{\partial G}{\partial c}$). The latter obtains because

$$\frac{\partial G}{\partial m^B} = u'(1, m^S, c)u''(1, m^B, v) - u''(0, M - m^B, c)u'(0, M - m^S, v)$$

and u is twice continuously differentiable and has a compact domain. For the former, we have

$$\frac{\partial F}{\partial c}=\frac{\partial u'(1,m^S,c)}{\partial c}X-u'(0,M-m^S,v)\left[\frac{\partial u(1,m^S,c)}{\partial c}-\frac{\partial u(0,M-m^B,c)}{\partial c}\right]$$

where $X=u(1,m^B,v)-u(0,M-m^S,v)$ and $u'(0,M-m^S,v)$ are bounded. Writing explicitly the arguments in m^B and m^S the remaining task is to show that $\frac{\partial u'(1,m^S(c,v),c)}{\partial c}$ and $\frac{\partial u(1,m^S(c,v),c)}{\partial c}-\frac{\partial u(0,M-m^B(c,v),c)}{\partial c}$ are small for small Δ. The difference is small because both its components are. Indeed, by the Implicit Function Theorem m^B and m^S are twice continuously differentiable and hence by taking L small we can ensure that $\left|\frac{\partial u(1,m^S(c,v),c)}{\partial c}-\frac{\partial u(1,m^S(c_0,v),c)}{\partial c}\right|<\frac{\varepsilon}{2}$ for all c,c_0 in the integration interval. At the same time, for $\Delta=\frac{\varepsilon}{4}$, the Δ bound on asymmetric information implies that that the absolute value of the integral of $\frac{\partial u(1,m^S(c_0,v),c)}{\partial c}$ over any integration interval is smaller than $\frac{\varepsilon}{2}$. Putting this together and requiring that $L\leq 1$, we conclude that the absolute value of the integral of $\frac{\partial u(1,m^S(c,v),c)}{\partial c}$ over any interval of length L or less is bounded by ε. An analogous argument obtains for the integrals over $\frac{\partial u(0,M-m^B(c,v),c)}{\partial c}$.

Finally we bound the integral over $\frac{\partial u'(1,m^S(c,v),c)}{\partial c}$ over c taken from an arbitrary interval of length L or less. This is equivalent to bounding the difference between $u'(1,m^S(c,v),c)$ and $u'(1,m^S(c',v),c')$ for $|c-c'|\leq L$. Equations (2) define the ratios

$$\frac{u'\left(1,m^S(c,v),c\right)}{u'(0,M-m^S(c,v),v)}=\frac{u'\left(0,M-m^B(c,v),c\right)}{u'(1,m^B(c,v),v)}=\frac{u\left(1,m^S,c\right)-u\left(0,M-m^B,c\right)}{u(1,m^B,v)-u(0,M-m^S,v)}.$$

By taking Δ small we ensure that for all c,v the right-hand-side ratio $\frac{u\left(1,m^S,c\right)-u\left(0,M-m^B,c\right)}{u(1,m^B,v)-u(0,M-m^S,v)}$ is within ε of $\frac{u\left(1,m^S,c^*\right)-u\left(0,M-m^B,c^*\right)}{u(1,m^B,v^*)-u(0,M-m^S,v^*)}$ because the nominator and denominator of this expression are positive and bounded away from 0. By the Implicit Function Theorem, $m^S(c,v)$ is twice continuously differentiable in c, and hence the continuous differentiability of u' in wealth implies that, for small Δ, we keep $u'\left(0,M-m^S(c,v),v\right)$ within ε of $u'\left(0,M-m^S(c^*,v),v\right)$. As $u'\left(1,m^S(c,v),c\right)$ is the product of the two quantities we just bounded, these bounds implies that by setting Δ small we can ensure that

$$\left|u'(1, m^S(c,v), c) - u'(1, m^S(c',v), c')\right| < \varepsilon$$

for all c and c' in the domain (and hence in particular for all c and c' such that $|c - c'| \leq L$).

C.4 Proof of Lemma 1

As a preparation, consider the PDEs

$$S_1(c,v)\frac{\partial}{\partial c}\pi(c,v) + S_2(c,v)\pi(c,v) = 0 \tag{17}$$

and

$$B_1(c,v)\frac{\partial}{\partial v}\pi(c,v) + B_2(c,v)\pi(c,v) = 0. \tag{18}$$

Considered separately, these equations are standard ODEs. Given the regularity assumptions we imposed, these ODEs have solutions for any initial conditions. Let us fix a solution $\Delta^b > 0$ to the first equation and a solution $\Delta^s > 0$ to the second that satisfy the initial conditions

$$\Delta^b(c^*, v) = \frac{1}{2} \quad \text{and} \quad \Delta^s(c, v^*) = \frac{1}{2}$$

for any c, v. For continuously differentiable functions $b(\cdot)$ and $s(\cdot)$, set

$$\pi(c,v) = b(v)\Delta^b(c,v) + s(c)\Delta^s(c,v)$$

and consider the second PDE from the lemma. The contribution of the second summand above to the equation's left-hand side is zero for each v, and thus it is zero in expectation. Thus, the second equation reduces to

$$\psi(v) = E_c\left[B_1(c,v)\frac{\partial}{\partial v}\left[b(v)\Delta^b(c,v)\right] + B_2(c,v)b(v)\Delta^b(c,v)\right], \tag{19}$$

where the right-hand side is equal to

$$E_c\left[B_1(c,v)\,\Delta^b(c,v)\right]b'(v)+E_c\left[B_1(c,v)\,\frac{\partial}{\partial v}\Delta^b(c,v)+B_2(c,v)\,\Delta^b(c,v)\right]b(v)\,.$$

Since $B_1\Delta^b>0$, this equation has solutions. Let b be one such solution satisfying the initial condition $b(v^*)\,\Delta^b(c^*,v^*)=\frac{1}{2}\pi^*$. Similarly, we can find a function s for which the first PDE from the lemma is satisfied and such that $s(c^*)\,\Delta^s(c^*,v^*)=\frac{1}{2}\pi^*$. Thus, for these two functions b and s, the function π defined above satisfies the system of PDEs from the lemma, as well as the initial condition.

C.5 The bound Δ on informational asymmetry and approximate flatness of π

To show the final (flatness) claim of the proof of Theorem 1 we establish the following property of the solution π constructed in the proof of Lemma 1: *for any $\delta>0$ there are $\overline{K},\underline{K}>0$, $\varepsilon>0$ and $L>0$ such that if (i) $|\phi|$, $|\psi|$, $|B_1|$, $|S_1|$, $|B_2|$, $|S_2|$ are bounded from above by $\overline{K}$ and $|B_1|$, $|S_1|$ are bounded from below by $\underline{K}$, (ii) for any v, the absolute value of the integral of $\frac{S_2(c,v)}{S_1(c,v)}$ over c in any interval of length at most L is bounded from above by ε, and (iii) for any c, the absolute value of the integral of $\frac{B_2(c,v)}{B_1(c,v)}$ over v in any interval of length at most L is bounded from above by ε, then the PDEs and the boundary condition of Lemma 1 have a solution π such that $|\pi(c,v)-\pi(c^*,v^*)|\leq\delta$ for any $c\in[\underline{c},\overline{c}]$ and $v\in[\underline{v},\overline{v}]$.*[35] The proof of Theorem 1 will be then complete because the analysis in Appendix C.3 shows that for any δ we can find such ε and L by taking Δ sufficiently low. To show the italicized implication we first bound the differences of functions Δ^s and Δ^b, and then bound the functions b and s.

Consider Δ^b which is the solution to (17). The coefficient in front of the derivative in this equation is non-zero everywhere and dividing this differential

[35]An inspection of the argument below shows that the claimed δ bound holds even if assumption (i) is substantially weakened.

equation by this coefficient we obtain

$$\frac{\partial}{\partial c}\Delta^b(c,v)+\frac{S_2(c,v)}{S_1(c,v)}\Delta^b(c,v)=0,$$

with solutions

$$\Delta^b(c,v)=k\exp\left(-\int_{\underline{c}}^{c}\frac{S_2(\tilde{c},v)}{S_1(\tilde{c},v)}d\tilde{c}\right)$$

parametrized by $k\in\mathbb{R}$. By assumption, for any v, the integral of $\frac{S_2(c,v)}{S_1(c,v)}$ over c in any interval of length at most L is bounded from above by ε; hence, for any $\hat{\delta}>0$, by taking ε small enough, we can guarantee that

$$\left|\Delta^b(c,v)-k\right|<\hat{\delta}$$

for any c and v in their respective bounded domains. An analogous argument allows us to ensure that

$$\left|\Delta^s(c,v)-k\right|<\hat{\delta}$$

for any c and v in their respective domains.

Consider now b, whose defining ODE is (as above)

$$E_c\left[B_1(c,v)\Delta^b(c,v)\right]b'(v)+E_c\left[B_1(c,v)\frac{\partial}{\partial v}\Delta^b(c,v)+B_2(c,v)\Delta^b(c,v)\right]b(v)=\psi(v).$$

As B_1 is continuously differentiable and non-zero, it is always strictly above 0 or else always strictly below 0. Consider the former case; the latter case is analogous. For $\hat{\delta}\in\left(0,\frac{k}{2}\right)$ the above bounds give us $\frac{3}{2}k>\Delta^b(c,v)>\frac{k}{2}>0$ for all c,v. As B_1 is continuous, its infimum on the domain is some $\underline{B}_1>0$, and thus, $E_c\left[B_1(c,v)\Delta^b(c,v)\right]>\frac{k}{2}\underline{B}_1>0$ for all v. The ODE coefficient $E_c\left[B_1(c,v)\frac{\partial}{\partial v}\Delta^b(c,v)+B_2(c,v)\Delta^b(c,v)\right]$ is bounded because B_2, Δ^b, and B_1 are bounded as is $\frac{\partial}{\partial v}\Delta^b(c,v)$; the latter is bounded in light of the exponential formula above because of the continuous differentiability of $S_2(\tilde{c},v)$ and $S_1(\tilde{c},v)$ in v and the strictly positive lower bound on S_1 established at the beginning of Appendix C.2. As $\psi(v)$ is bounded by $\overline{K}$, we infer that there is a solution b that is uniformly bounded from above and the bound does not depend on $\hat{\delta} \in \left(0, \frac{k}{2}\right)$. Analogously, s is uniformly bounded from above.

Taking $\hat{\delta}$ sufficiently small and combining the above bounds, gives us $|\pi(c, v) - \pi(c^*, v^*)| \leq \delta$, thus proving the flatness claim.

D Derivation of the Efficient Contract for the Case of Separable Utilities (Including Equations (8)–(11))

In the separable case, the equations given in (2) that define the money levels $m^S(c,v)$ and $m^B(c,v)$ associated with the points $S(c,v)$ and $B(c,v)$ in Figures 1-3, reduce to

$$\frac{\frac{\partial}{\partial m}U(m^S)}{\frac{\partial}{\partial m}U(M-m^S)} = \frac{\frac{\partial}{\partial m}U(M-m^B)}{\frac{\partial}{\partial m}U(m^B)} \tag{20}$$

and

$$c + U(m^S) = U(M-m^B) + \frac{\frac{\partial}{\partial m}U(m^S)}{\frac{\partial}{\partial m}U(M-m^S)} * [v + U(m^B) - U(M-m^S)]. \tag{21}$$

Equation (20) implies (8). To see this, suppose $m^S > M - m^B$. Then, $m^B > M - m^S$ and since U is strictly increasing in m, the left-hand side of (20) would be strictly greater than the right-hand side, contradicting (8). The reverse contradiction occurs if we assume $m^S < M - m^B$. In other words, in the optimal contract for the separable case, each player has equal money in each state.

Substituting (8) into (21) yields (9). Given any pair (c,v), equation (9) uniquely defines $m^S(c,v)$, and $m^B(c,v)$ is then given by (8).

Equation (8) implies that $\frac{\partial}{\partial \hat{c}}u(1, m^S(\hat{c},v); c) = \frac{\partial}{\partial \hat{c}}u(0, M - m^B(\hat{c},v); c) = \frac{\partial}{\partial \hat{c}}U(m^S(\hat{c},v))$ and that $\frac{\partial}{\partial \hat{v}}u(0, M - m^S(c,\hat{v}); v) = \frac{\partial}{\partial \hat{v}}u(1, m^B(c,\hat{v}); v) = \frac{\partial}{\partial \hat{v}}U(m^B(c,\hat{v}))$. Thus, the first-order equations we derived in Section 3.1 take the form (10) and (11).

The second-order conditions of the incentive-compatibility analysis are satisfied in the separable case. Indeed, in light of Appendix C.1, it is sufficient to verify condition (1), which is satisfied in the separable case because $\frac{\partial log(\theta)}{\partial \theta} = \frac{1}{\theta} > 0$ and $\frac{\partial log(U'(m))}{\partial \theta} = 0$.

E Proof of Theorem 2

The ratio of seller's and buyer's marginal utility from having the good is equal to the ratio of their types. Because the types come from a compact subset of $(0, \infty)^2$, we may assume that the types belong to $[\theta_L, \theta_H]$ for some $\theta_H > \theta_L > 0$. Let $\Delta_{SB} > \frac{\theta_H}{\Theta_L}$; in particular, for every profile of types this ratio belongs to the interval $\left[\frac{1}{\Delta_{SB}}, \Delta_{SB}\right]$.

First consider the case that the buyer initially has less money than the seller. Observe that were the money split half-half, our assumptions imply that seller/buyer ratio of marginal utilities of money is equal to 1. Thus, for sufficiently high Δ, the assumption that $|U''|/U'$ is above Δ at almost all relevant money levels implies that the seller/buyer ratio of marginal utilities of money is smaller than $\frac{1}{\Delta_{SB}}$ at the initial money holdings (and after every positive money transfer from the buyer to the seller). The preceding observation on utility ratios thus implies that no-trade is Pareto efficient.

Second consider the complementary case that the buyer initially has more money than the seller. Let m_B denote the initial money holdings of the buyer and m_S the initial money holdings of the seller. Let us take a parameter $\gamma > 0$ and consider an auxiliary problem in which types $\tilde{\theta}$ belong to $\left[\frac{\theta_H+\theta_L}{2} - \frac{\theta_H-\theta_L}{2\gamma}, \frac{\theta_H+\theta_L}{2} + \frac{\theta_H-\theta_L}{2\gamma}\right]$ and the utility of money is

$$\tilde{U}(m) = U\left(\frac{m_S + m_B}{2} + \frac{1}{\gamma}\left(m - \frac{m_S + m_B}{2}\right)\right).$$

In particular, $U(m) = \tilde{U}\left(\frac{m_S+m_B}{2} + \gamma\left(m - \frac{m_S+m_B}{2}\right)\right)$. For fixed $\tilde{U}$, Theorem 1 implies that if γ is sufficiently large then in the auxiliary problem, efficient trade is possible except at the critical initial money distribution. This exception never obtains for large γ because (a) the proof of Theorem 1 tells us that the critical money distribution is such that the ratio of buyer and seller types exactly equal the ratio of marginal utilities of money at initial money holdings; and (b) as we increase γ while keeping m_S, m_B, and $\tilde{U}$ constant, the ratio of marginal utilities of money stays constant and different from 1, and the ratio of types is arbitrarily close to 1.

To close the proof we build on our analysis of incentive compatible mechanisms in Theorem 1 to conclude that we obtain efficient trade if γ is sufficiently high and $\tilde{U}$ satisfies $|\tilde{U}''|/\tilde{U}' \geq \tilde{\Delta}$ almost everywhere for sufficiently large $\tilde{\Delta} > 0$. From Section D of the Appendix and the constructive proof of Lemma 1, we know that a candidate efficient trade solution is given by the efficient money shares $m^S(c,v)$ and $m^B(c,v)$ given by

$$c\tilde{U}'\left(M - m^S\right) = v\tilde{U}'\left(m^S\right) \tag{22}$$

and $m^S + m^B = M$, where M is the total money amount.

Claim. Equation (22) has a solution provided $|\tilde{U}''|/\tilde{U}' \geq \tilde{\Delta}$ almost everywhere for sufficiently large $\tilde{\Delta} > 0$.

To prove this claim, note that we can rewrite (22) as

$$cU'\left(\frac{M}{2} + \frac{1}{\gamma}\left(M - m^S - \frac{M}{2}\right)\right) = vU'\left(\frac{M}{2} + \frac{1}{\gamma}\left(m^S - \frac{M}{2}\right)\right)$$

We use the Intermediate Value Theorem to show that this equation has a solution. We first show that if $m^S = 0$, then LHS < RHS. At $m^S = 0$ we have

$$\text{LHS} = cU'\left(\frac{M}{2}\left(1 + \frac{1}{\gamma}\right)\right) \text{ and RHS} = vU'\left(\frac{M}{2}\left(1 - \frac{1}{\gamma}\right)\right).$$

By strict concavity of U, we know $U'\left(\frac{M}{2}\left(1+\frac{1}{\gamma}\right)\right) < U'\left(\frac{M}{2}\left(1-\frac{1}{\gamma}\right)\right)$. Hence, if $c < v$ we are done. On the other hand, if $c > v$, then we use the fact that for sufficiently large $\Delta > 0$, $|U''|/U' \geq \Delta$ almost everywhere implies that $\frac{U'(m_s)}{U'(m_b)} > \frac{\theta_H}{\theta_L}$ for all $m_b > m_s$. This condition ensures that $\frac{U'\left(\frac{M}{2}\left(1-\frac{1}{\gamma}\right)\right)}{U'\left(\frac{M}{2}\left(1+\frac{1}{\gamma}\right)\right)} > \frac{c}{v}$ for all γ and for all c and v such that $c > v$, proving that LHS < RHS at $m^S = 0$.

Second, we show that if $m^S = M$, then LHS > RHS. At $m^S = M$ we have

$$\text{LHS} = cU'\left(\frac{M}{2}\left(1 - \frac{1}{\gamma}\right)\right) \text{ and } \text{RHS} = vU'\left(\frac{M}{2}\left(1 + \frac{1}{\gamma}\right)\right).$$

Similar to before, by strict concavity of U, if $v < c$ we are done. On the other hand, if $v > c$, then we use the fact that for sufficiently large $\Delta > 0$, $|U''|/U' \geq \Delta$ almost everywhere implies $\frac{U'(m_s)}{U'(m_b)} < \frac{\theta_L}{\theta_H}$ for almost all $m_b < m_s$. This condition ensures that $\frac{U'\left(\frac{M}{2}\left(1+\frac{1}{\gamma}\right)\right)}{U'\left(\frac{M}{2}\left(1-\frac{1}{\gamma}\right)\right)} < \frac{c}{v}$ for all γ and for all c and v such that $v > c$, proving that LHS > RHS at $m^S = M$. The existence of a solution m^S with $0 \leq m^S \leq M$ follows from continuity of U and the Intermediate Value Theorem.

Having proven the claim, we return to the proof of Theorem 2. Note that $m^S > m^B$ if and only if $c > v$. As we grow γ keeping $\tilde{U}$ constant, money levels m^S and m^B both converge towards $\frac{M}{2}$ for all c, v in the support, and differentiating equation (22) with respect to c gives us

$$\frac{\partial m^S}{\partial c} = \frac{\tilde{U}'\left(M - m^S\right)}{c\tilde{U}''\left(M - m^S\right) + v\tilde{U}''\left(m^S\right)}.$$

For large $\tilde{\Delta}$, this implies that almost everywhere $\frac{\partial m^S}{\partial c}$ is small in absolute value; it belongs to $\left(-\frac{1}{c\tilde{\Delta}}, 0\right)$. An analogous argument works for the other partial of m^S and for the partials of m^B. Furthermore, for large γ the support of types is small and the two differential equations are roughly symmetric ensuring that efficient money levels m^S and m^B are both close $\frac{M}{2}$.

In line with the analysis of Lemma 1, $\Delta^b(c, v) = 1$ is a solution to the auxiliary equation $c\frac{\partial \pi(c,v)}{\partial c} = 0$, and $\Delta^s(c, v) = 1$ is a solution to the analogous second auxiliary equation. For the probability of passing the good, we can hence take $\pi(c, v) = C + b(v) + s(c)$, where C is an arbitrary constant set so as to ensure $\pi \in [0, 1]$, b is a solution to the equation $\mathbb{E}_c\left\{b'(v)\, v + \frac{\partial}{\partial v}\tilde{U} \circ m^B(c, v)\right\} = 0$, e.g.

$$b(v) = \int_{\underline{v}}^{v} -\frac{\mathbb{E}_c\left\{\frac{\partial}{\partial v}\tilde{U} \circ m^B(c, \tilde{v})\right\}}{\tilde{v}} d\tilde{v},$$

and s is a solution to the analogous equation, e.g.

$$s(c) = \int_{\underline{c}}^{c} -\frac{\mathbb{E}_v \left\{ \frac{\partial}{\partial v} \tilde{U} \circ m^B (\tilde{c}, v) \right\}}{\tilde{c}} d\tilde{c}.$$

Setting $\theta^* = \frac{\theta_H + \theta_L}{2}$, we may observe that $\frac{1}{\tilde{c}} \approx \frac{1}{\theta^*}$ and $\tilde{U}' \approx \tilde{U}' \left(\frac{M}{2}\right)$ are both roughly constant for large γ, and our bounds on the partials of m^S, m^B imply that s and b can be taken to be close to 0 for large $\tilde{\Delta}$. Thus, for large γ and $\tilde{\Delta}$, the efficient probability transfers can be implemented in an incentive compatible way (as guaranteed by the analysis of Lemma 1) with a nearly constant probability of object transfer. In particular, for any $C \in (0,1)$ the probabilities $\pi(c, v)$ are indeed in $[0,1]$ for all c, v as soon as γ and $\tilde{\Delta}$ are sufficiently large.

The last thing to check is that by setting C properly we can ensure the individual rationality of both the buyer and the seller. Notice that for C close to 0, the seller's individual rationality is satisfied. The post-trade total welfare is strictly larger than it is at the status quo because the post-trade welfare differs vis-a-vis the status quo primarily by the shift of money from the buyer with relatively low marginal value of money to the seller with relatively high marginal value of money. As we increase C, we shift the surplus from the seller to the buyer while the total welfare stays constant by construction. There are two cases:

(a) For C close to 1 the individual rationality of the buyer θ^* is strictly satisfied. Then, for high γ individual rationality is satisfied for all buyer types. Consequently, there is a range of C in which both buyer and seller's individual rationality is satisfied for all types, and the theorem obtains.

(b) For $C = 1$ the buyer's individual rationality fails or is exactly satisfied for buyer's type θ^*. The welfare gain of the sure trade with money levels m^S and m^B is bounded away from zero for all buyer and seller types, and the utility gain of seller types are also bounded away from zero. (Note that we do not claim that sure trade with money levels m^S and m^B is incentive compatible; it is not). We can then find a monetary transfer T that does not depend on the type configuration, that is smaller than the transfer $M^B - m^B$ (where M^B denotes the initial money holdings of the buyer) for all buyer and seller types, and such that sure trade accompanied by transfer T is Pareto efficient (where Pareto efficiency follows from $T < M^B - m^B(c, v)$ for all types).

F Proof of Theorem 3

First notice that without loss of generality we can normalize $M = 1$. Consider the mechanism that assigns the good and all the money to the seller with probability m_s (the initial seller's share of the aggregate money holding) and with the complementary probability $m_b = 1 - m_s$, it assigns the good and all the money to the buyer. As we do not need to elicit the types, strategy-proofness is for free. The individual rationality for the seller follows directly from the inequality assumed in the theorem by setting $\theta = c$ and $\lambda = \frac{m_s}{M}$ and recognizing that the inequality is satisfied also for $\lambda \in \{0, 1\}$. The individual rationality for the buyer follows from the inequality assumed in the theorem by setting $\theta = v$ and $\lambda = \frac{m_b}{M}$ and recognizing that $u(1, \lambda M, \theta) \geq u(0, \lambda M, \theta)$ for any $\lambda \in [0, 1]$. Because the latter inequality is strict, the buyer strictly gains from trade.

To prove that the outcomes are ex-post Pareto efficient, consider any outcome in which the seller gets the object and money y with probability π and no object and money z with probability $1 - \pi$; while the buyer gets the reminder in both states. The seller's expected utility is

$$u^S = \pi u(1, y, v) + (1 - \pi) u(0, z, v),$$

and using the assumption of the theorem we can bound it from above by a convex combination of the outcomes of our mechanism:

$$\begin{aligned} u^S &\leq \pi (yu(1, M, v) + (1 - y) u(0, 0, v)) + (1 - \pi)(zu(1, M, v) + (1 - z) u(0, 0, v)) \\ &\leq (\pi y + (1 - \pi) z) u(1, M, v) + (\pi (1 - y) + (1 - \pi)(1 - z)) u(0, 0, v). \end{aligned}$$

Furthermore, this convex combination also bounds buyer's utility u^B from above:

$$u^B \leq (\pi (1-y) + (1-\pi)(1-z))\, u(1, M, c) + (\pi y + (1-\pi) z)\, u(0, 0, c) .$$

Thus, the mechanism is Pareto efficient.